Colored HOMFLY and WZW Models

Ref. 1407.5643 J.G. Jockers "A note on colored HOMFLY polynomials for hyperbolic knots from WZW model"
1401.5095

* Chern-Simons theory with $SU(N)$ and level k in S^3

$$S = \frac{k}{4\pi}\int_{S^3} \mathrm{Tr}_R \left(A\wedge dA + \frac{2}{3} A\wedge A\wedge A \right)$$

A: $SU(N)$ connection in repr. R

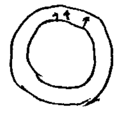
k: coupling, has to be integer

- No metric, topological gauge field theory

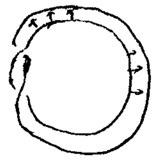
Observable: Wilson loop $S^1 \hookrightarrow S^3$ (K)

$$W_R^{SU(N)}(K) = \langle \mathrm{Tr}_R P \exp \oint_K A \rangle$$

- topological invariant of knot
- For $W_R^{SU(N)}(K)$ to be well-defined. framing of knot: global section of normal bundle to K



$f=0$



$f=1$

the linking number of K and its companion K' obtained by deforming K along the direction of the global section.

- framing transformation:

$$W_R^G(K) \longmapsto e^{2\pi i h_R \cdot f} W_R^G(K)$$

$$h_R = \sum \frac{(T^a)^2}{2(k+g)} = \frac{C_R}{k+g}$$ conformal weight of WZW primary

For $SU(N)$ $\left(C_R = \frac{1}{2}\left[N\cdot \ell + \kappa_R - \frac{\ell^2}{N}\right] \right)$

$$W_R^{SU(N)}(K) \longmapsto q^{-\frac{\ell^2}{2N}\delta f} \lambda^{\frac{1}{2}\ell \delta f} q^{\frac{1}{2}\kappa_R \delta f} W_R^{SU(N)}(K)$$

$q = \exp\left(\frac{2\pi i}{k+N}\right)$, $\lambda = q^N$, $\kappa_R = \ell + \sum (l_i^2 - 2i\, l_i)$

* Witten's discovery

$$H_R(K) \equiv q^{\frac{\ell^2}{2N}\delta f} W_R^{SU(N)}(K)$$: ① function of q, λ only (no N), in any framing

$$H_R \longmapsto \lambda^{\pm \ell \delta f} q^{\frac{1}{2}\kappa_R \delta f} H_R$$

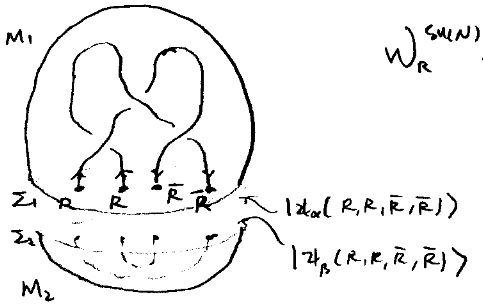
② normalization = colored HOMFLY $\left.\frac{H_R(K)}{H_R(0)}\right|_{\substack{q\to q^2 \\ \lambda \to \lambda^2}}$

* $\{ H_R(K),\ R \text{ all irrep} \}$ → complete knot invariant → brane partition function

status: K: torus knot
R: symmetric

3_1

* Witten's insight.



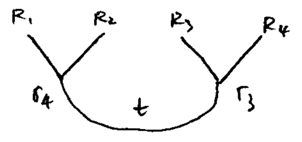
$$W_R^{SU(N)}(K) = \langle \psi_\beta | \psi_\alpha \rangle$$

Hilbert space on Σ with n punctures colored in $R_1 \cdots R_n$ $(R, \bar{R})$

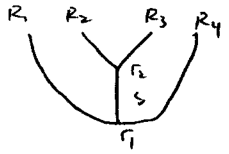
$\Downarrow$ S

Vector space of n-pt conformal blocks with the n fields being primaries in $R_1 \cdots R_n$

Four punctures.

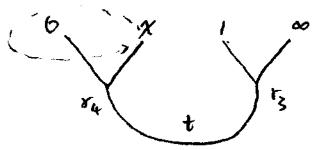


$|\psi^{(1)}_{t, r_3 r_4}(R_1 R_2 R_3 R_4)\rangle$



$|\psi^{(2)}_{s, r_1 r_2}(R_1 R_2 R_3 R_4)\rangle$

* Closer look

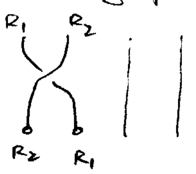


conformal block: $\mathcal{F}^{21}_{34}(t|x) = x^{h_t - h_1 - h_2} \cdot P(x)$

monodromy: $\mathcal{F}^{21}_{34} \longmapsto q^{C_t - C_{R_1} - C_{R_2}} \mathcal{F}^{21}_{34}$ $\quad e^{2\pi i \cdot h_R} = e^{2\pi i \frac{C_R}{k+N}} = q^{C_R}$

$x \mapsto e^{2\pi i} x$

braiding operator. move x through half of the loop.



side braiding. basis 1 (t-channel)

central braiding. basis 2 (s-channel)

$$\lambda = \pm q^{\pm(C_{R_i} + C_{R_j} - C_{R_k})/2}$$

3j-phases

$+$: right-handed braiding

$-$: left-handed braiding

i, j: two legs,

k: intermediate

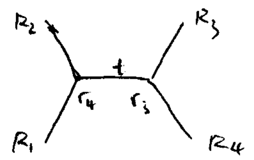
3j-phase:

$$\langle r R_k m_k | R_i m_i\, R_j m_j \rangle = \{ R_i R_j \bar{R}_k, r \} \langle r R_k m_k | R_j m_j, R_i m_i \rangle$$

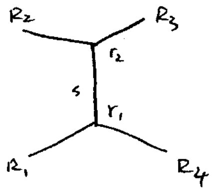
a triad: on equal footing

- $R_i = R_j$, R_k being in the symmetric or anti-symmetric tensor-product; free.
- $R_k = \mathbb{1}$: 3j-phase $\{R_i\} \equiv \{R_i \bar{R}_i \mathbb{1}\}$

- Basis transformation: given by fusion rules.



$$= \sum_{\substack{s \\ r_1 r_2}} a^{t, r_3 r_4}_{s, r_1 r_2} \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$$



- difficult: (1 or only with $\square$, $\square\square$, symmetric reprs) $\rightsquigarrow$ Core problem: a CFT problem

* Direct computation of $W_R^{SU}(K)$

- Project knot to plane like in previous page
- Cut S^3 together with projection. simple bottom $|\psi_\beta\rangle$ ✓, rest in upper $|\psi_\alpha\rangle$
- $|\psi_\alpha\rangle$ start with simple-top, apply braiding operators. b_1: use basis 1. b_2 use basis 2
  switching braiding $\longrightarrow$ fusion matrices
- $\langle\psi_\alpha|\psi_\beta\rangle$; framing = # (undercrossing − over crossing) ╳ − ╳

Quantum 6j-symbols

* Quantum group $U_q(\mathfrak{g})$: q-deformation of universal enveloping algebra $U(\mathfrak{g})$

e.g. sl_2. E, F, H $\quad [H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H$

$U sl_2$: E, F, H $\quad HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H$ $\quad$ unital associative algebra over $\mathbb{C}$.

$U_q sl_2$. E, F, K, K^{-1} $\quad K K^{-1} = K^{-1} K = 1$

$$KE = qEK, \quad KF = q^{-1} FK, \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}$$

deformation. $K = q^{H/2} = e^{\hbar H/2}$

$$KE = e^{\hbar H/2} E = E e^{\hbar (H+2)/2} = q E K$$

$$KF = e^{\hbar H/2} F = F e^{\hbar (H-2)/2} = q^{-1} F K$$

$$\frac{K - K^{-1}}{q^{1/2} - q^{-1/2}} = [H] \xrightarrow{q \to 1} H = EF - FE$$

- Modules over $U_q sl_N$, reps

  q not being root of unit or q being unit of root $q^n = 1, n \gg 1$ $\rightsquigarrow$ 1:1 irreps of sl_N.

- Connection with WZW models

  ① WZW primaries $\xleftrightarrow{1:1}$ irreps/modules of $U_q sl_N$. $q = \exp \frac{2\pi i}{k+N}$

  fusion matrices $\xleftrightarrow{1:1}$ recoupling coefs.

* Recoupling coef's

Gives isomorphism. $\alpha: (V_1^{\lambda_1} \otimes V_2^{\lambda_2}) \otimes V_3^{\lambda_3} \xrightarrow{\sim} V_1^{\lambda_1} \otimes (V_2^{\lambda_2} \otimes V_3^{\lambda_3})$

$\lambda_1 \otimes \lambda_2 \otimes \lambda_3 \longrightarrow \lambda$, vector m

$|\lambda m\rangle$ appears repeatedly with different λ_{12}

label $|\lambda, m\rangle$ by λ_{12}, r_{12}, r

$|(\lambda_1\lambda_2) \underline{r_{12}}\, \underline{\lambda_{12}}, \lambda_3, \underline{r}\lambda m\rangle_q$

orthonormal basis

$|\lambda_1 (\lambda_2\lambda_3) r_{23} \lambda_{23}, r'\lambda m\rangle_q$

orthonormal basis

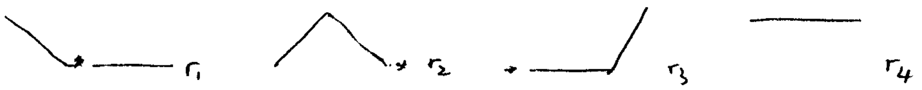
basis transformation.

$$|(\lambda_1\lambda_2) \underline{r_{12}}\, \underline{\lambda_{12}}, \underline{\lambda_3}, \underline{r}, \lambda m\rangle_q = \sum_{\substack{r_{23}, \lambda_{23} \\ r'}} |\lambda_1 (\lambda_2\lambda_3) \underline{r_{23}}\, \underline{\lambda_{23}}, \underline{r'}\lambda m\rangle_q \cdot \underline{\langle \lambda_1 (\lambda_2\lambda_3) r_{23} \lambda_{23}, r'\lambda \,|\, (\lambda_1\lambda_2) r_{12} \lambda_{12}, \lambda_3, r\lambda\rangle_q}$$

Normalization: quantum 6j-symbol

$$_q\begin{Bmatrix} \lambda_1 & \lambda_2 & \bar{\lambda}_{12} \\ \lambda_3 & \lambda & \lambda_{23} \end{Bmatrix}_{r' r_{23} r\, r_{12}} = \frac{\{\lambda_2\}\{\lambda_1\lambda_2\bar{\lambda}_{12} r_{12}\}\{\lambda_{12}\lambda_3\bar{\lambda} r\}}{\sqrt{\dim_q \lambda_{12} \dim_q \lambda_{23}}} \langle \lambda_1 (\lambda_2\lambda_3) r_{23} \lambda_{23}; r'\lambda \,|\, (\lambda_1\lambda_2) r_{12} \lambda_{12}, \lambda_3, r\lambda\rangle_q$$

Four triads



* Symmetry properties

(i) Tetrahedral symmetry

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} = \begin{Bmatrix} \lambda_2 & \lambda_3 & \lambda_1 \\ \mu_2 & \mu_3 & \mu_1 \end{Bmatrix}_{r_2 r_3 r_1 r_4} = \{\mu_1\}\{\mu_2\}\{\mu_3\}\{\lambda_1 \bar{\mu}_2 \mu_3 r_1\}\{\mu_1 \lambda_2 \bar{\mu}_3 r_2\} \{\bar{\mu}_1 \mu_2 \lambda_3 r_3\}\{\lambda_1\lambda_2\lambda_3 r_4\} \begin{Bmatrix} \lambda_2 & \lambda_1 & \lambda_3 \\ \bar{\mu}_2 & \bar{\mu}_1 & \bar{\mu}_3 \end{Bmatrix}_{r_2 r_1 r_3 r_4}$$

$$= \begin{Bmatrix} \bar{\lambda}_1 & \mu_2 & \bar{\mu}_3 \\ \bar{\mu}_1 & \lambda_2 & \bar{\lambda}_3 \end{Bmatrix}_{r_4 r_3 r_2 r_1}$$

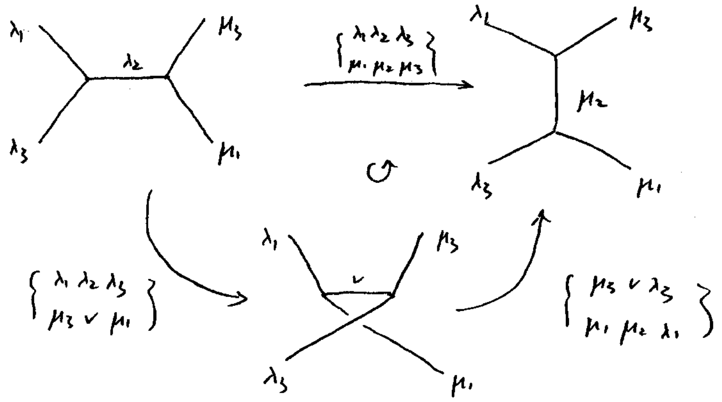
(ii) $$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} = \begin{Bmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}_3 \\ \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_3 \end{Bmatrix}^*_{r_1 r_2 r_3 r_4}$$

(iii) Unitarity $$\sum_{\mu_3 r_1 r_2} |\lambda_3| |\mu_3| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3' \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r_1 r_2 r_3' r_4'} = \delta_{\lambda_3\lambda_3'} \delta_{r_3 r_3'} \delta_{r_4 r_4'}$$

(iv) Generalized Racah backcoupling rule

$$q^{(C_{\lambda_1} + C_{\mu_1} + C_{\lambda_3} + C_{\mu_3})/2} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$$

$$= \sum_{\nu r s} q^{(C_\nu + C_{\lambda_2} + C_{\mu_2})/2} |\nu| \{\lambda_3\}\{\lambda_1 \bar{\mu}_2 \mu_3 r_1\}\{\mu_1 \lambda_2 \bar{\mu}_3 r_2\}\{\bar{\lambda}_1 \mu_1 \nu r\} \begin{Bmatrix} \mu_3 & \nu & \lambda_3 \\ \mu_1 & \mu_2 & \lambda_1 \end{Bmatrix}_{r_1 r\, r_3 s} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\mu}_3 & \nu & \bar{\mu}_1 \end{Bmatrix}_{r r_2 s r_4}$$



(sloppy on $\bar{\ }$ and r_i)

(iv) Pentagon relation (Biedenharn-Elliott sum rule)

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} = \sum_{\substack{\lambda, \nu_3 \\ t_1 t_2 t_3 s_1 s_2}} |\lambda_3||\nu_3||\lambda| \, \{\lambda_4\}\{\nu_1\}\cdot$$

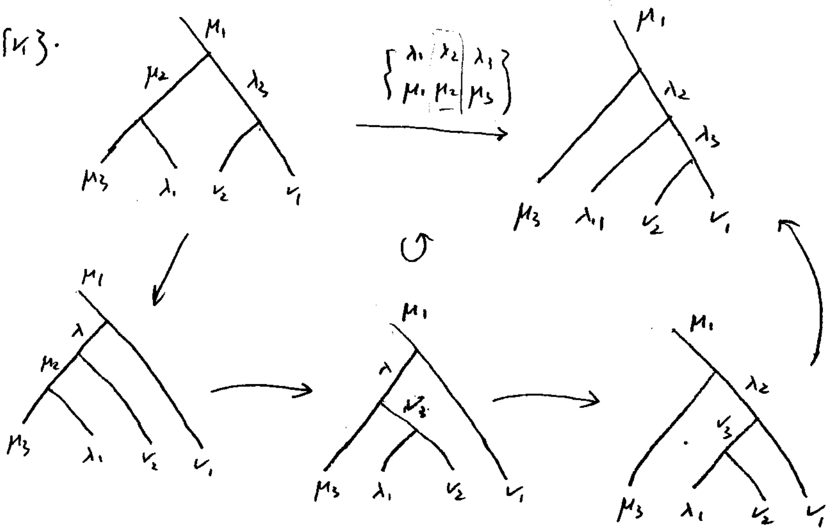
$$\{\lambda_1 \bar{\mu}_2 \mu_3 r_1\}\{\mu_1 \lambda_2 \bar{\mu}_3 r_2\}\{\bar{\mu}_1 \mu_2 \lambda_3 r_3\}$$

$$\{\lambda \bar{\mu}_1 \nu_1 t_1\}\{\lambda \bar{\mu}_2 \nu_2 t_2\}\{\lambda \bar{\mu}_3 \nu_3 t_3\}$$

$$\begin{Bmatrix} \underline{\nu_2} & \bar{\mu}_2 & \lambda \\ \mu_3 & \nu_3 & \lambda_1 \end{Bmatrix}_{s_1 r_1 t_3 t_2} \begin{Bmatrix} \nu_3 & \mu_3 & \lambda \\ \mu_1 & \underline{\nu_1} & \lambda_2 \end{Bmatrix}_{s_2 r_2 t_1 t_3}$$

$$\begin{Bmatrix} \underline{\nu_1} & \bar{\mu}_1 & \lambda \\ \mu_2 & \underline{\nu_2} & \lambda_3 \end{Bmatrix}_{s_3 r_3 t_2 t_1} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \underline{\nu_1} & \underline{\nu_2} & \nu_3 \end{Bmatrix}_{s_1 s_2 s_3 r_2}$$

ν_3 λ sum over $\underline{\nu_1}$ $\underline{\nu_2}$ arbitrary



* Composite labelling: independent of rank of group.

Recursive definition: $\square = (1;0)$, $\bar{\square} = (0;1)$

$(\mu_1;\nu_1) \otimes (\mu_2;\nu_2)$ = Step 1: $U(N)$ tensor product, $N \gg 1$ #(boxes) preserving

partition

$$\mu_1 \otimes \mu_2 = \sum_r \rho_r$$

$$\nu_1 \otimes \nu_2 = \sum_s \sigma_s$$

Step 2: index contraction

$|\rho_r, \sigma_s)$: pair-wise subtracting boxes from ρ_r, σ_s s.t. the leftovers from ρ_r, σ_s are both standard tableaux, and if two boxes subtracted from ρ_r are in a row, then their partners in σ_s cannot be in a column, and vice-versa.

$(1,0)\otimes(0,1) \to (0,0)\oplus(1,1)$

$(2,0)\otimes(0,1^2) \to (2,1^2)\oplus(1,1)$

$$(\mu_1;\nu_1)\otimes(\mu_2;\nu_2) = \bigoplus_{r,s} \bigoplus_{\zeta_{rs}} (\rho_r/\zeta_{rs} ; \sigma_s/\zeta_{rs})$$

$$\begin{pmatrix} \mu\otimes\nu = \sum_\lambda N^\lambda_{\mu\nu} \cdot \lambda \\ \lambda/\mu = \sum_\nu N^\lambda_{\mu\nu}\, \nu \end{pmatrix}$$

Relation to λ: $(\mu,\nu) = (\overbrace{\mu_1\, \mu_2 \cdots \mu_p, 0, \cdots 0, -\nu_3, \cdots -\nu_2, -\nu_1}^{N})$

$= (\lambda_1\, \lambda_2 \cdots \lambda_N)$ shift by the same integer

$= \lambda$

~~Advantage~~: indep of N, $(\mu,\nu)^* = (\nu,\mu)$

Bootstrap

Idea. with these symmetry properties one only need input of [quantum dimensions and fusion rules].

* Classification. trivial, primitive, non-primitive

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2^* & \lambda_1 & 0 \end{Bmatrix}_{00rs} = \frac{\{\lambda_1 \lambda_2 \lambda_3 r\}}{\sqrt{\dim_q \lambda_1 \dim_q \lambda_2}} \delta_{rs}$$

* Non-primitive $\longrightarrow$ primitive

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$$

λ_2 : smallest number of boxes

Pentagon : $\nu_1 = (1:0)$ or $(0:1)$, ν_2 one fewer box than λ_3 s.t. $(\bar{\nu}_1 \nu_2 \lambda_3)$ ✓

$$= \sum c \cdot \begin{Bmatrix} \lambda_1 & \nu_3 & \bar{\nu}_2 \\ \lambda & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_5' r_2' r_1'}$$

primitives

#(boxes) of the smallest repr. cannot exceed that of ν_2

* Primitive $\longrightarrow$ Core (Pentagon + backcoupling [Searle, 1988])

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \epsilon & \mu_3 \end{Bmatrix}_{0 r_2 0 r_4} \qquad \begin{Bmatrix} \lambda_1 & \lambda_2 & \epsilon_1 \\ \mu_1 & \mu_2 & \epsilon_2 \end{Bmatrix}_{0000}$$

type II (core) — type IV

$\lambda_1 \geq \lambda_2 \geq \lambda_3$, $\lambda_3 > \mu_1$ or $\lambda_1 > \mu_3$ — all primitive triads

* Solving type IV

- Suppose ϵ_1 and ϵ_2 are conjugate to each other
- $\lambda_1 = \mu_1$ Apply backcoupling rule

$$g^{-} \begin{Bmatrix} \lambda_1 & \lambda_2 & \epsilon_1 \\ \mu_1 & \mu_2 & \epsilon_2 \end{Bmatrix} = \sum g^{(C_\nu + C_{\lambda_2} + C_{\mu_2})/2} \begin{Bmatrix} \epsilon_2 & \nu & \epsilon_1 \\ \mu_1 & \mu_2 & \bar{\lambda}_1 \end{Bmatrix} \begin{Bmatrix} \lambda_1 & \lambda_2 & \epsilon_1 \\ \bar{\epsilon}_2 & \nu & \bar{\mu}_1 \end{Bmatrix}$$

ν : $(0;0)$ or $(1;1)$

Rescale to eliminate coef. before $\nu = (1:1)$ / a second identity $q \to q^{-1}$ / difference : no $\nu = (1:1)$

- $\lambda_1 \neq \mu_1$ the repr at $(2,1)$ is either unique (μ_1) or can only be either μ_1 or λ_1. Unitarity + free phase.

* Solving type II (core)

No universal algorithm like type IV. Case by case.

- Some repr. being unique e.g. $\begin{Bmatrix} 4:0 & 0:2 & 0:2 \\ 0:1 & 1:0 & 0:3 \end{Bmatrix}$
- Descendable at pos (1,1) or (1,3). e.g. $\begin{Bmatrix} 3:0 & 0:3 & 1:1 \\ 1:0 & 0:1 & 0:2 \end{Bmatrix}$

  The repr. at (1,3) is either λ_3 or something with fewer boxes.

- $\begin{Bmatrix} \lambda_1 & \lambda_2 & 2:0 \\ 1:0 & 0:1 & \mu_3 \end{Bmatrix}$ $\begin{Bmatrix} \lambda_1 & \lambda_2 & 1^2:0 \\ 1:0 & 0:1 & \mu_3 \end{Bmatrix}$

  μ_3 is either unique, or two options

  $\begin{Bmatrix} 31:0 & 0:2 & 0:1^2 \\ 0:1 & 1:0 & \mu_3 \end{Bmatrix}$ $\mu_3 = \begin{matrix} 0:3 \\ 0:21 \end{matrix}$

* Multiplicity
  - What if for a type II r_4 can be non-trivial?
  - Suppose $\lambda_1 \otimes \lambda_2 = m\,\bar{\lambda}_3$ so that $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \epsilon & \mu_3 \end{Bmatrix}_{0\,0\,0\,r_4}$ $r_4 = 0, 1, \cdots m-1$, and suppose

    $\sum_{r_4} \left| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \epsilon & \mu_3 \end{Bmatrix} \right|^2$ known for different μ_3 (unitarity sum) as well as $\sum_{r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \epsilon & \mu_3 \end{Bmatrix} \cdot \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \epsilon & \mu_3' \end{Bmatrix}^*$ (orthogonality sum)

    with the help of unitarity.
  - $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \epsilon & \mu_3 \end{Bmatrix}_{r_4}$ not fixed: $SU(m)$ freedom: $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \epsilon & \mu_3 \end{Bmatrix}^{\text{new}}_{r_4} = \sum_{r_4'} M_{r_4 r_4'} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \epsilon & \mu_3 \end{Bmatrix}^{\text{old}}_{r_4'}$

  Multiplicity separation scheme

r_4 \ μ_3	1	2	...
0	*	-	
1	0	*	
2	0	0	
⋮	⋮	⋮	
m−1	0	0	

  1. Set higher r_4 to be zero, and the absolute value of the lowest r_4 left is fixed by unitarity sum. Choose phase freely.
  2. − fixed by orthogonality sum.

     * fixed by unitarity sum.
  3. DOF used in this scheme: at most $\begin{matrix} 2(m-1) + 2(m-2) + \cdots + 1\cdot 2 \\ +1 + \cdots + 1 \end{matrix} = \begin{matrix} m^2-1 \\ +1 \end{matrix}$ of $SU(m)$!

- The goal

  $\begin{Bmatrix} R & \bar{R} & \rho_1 \\ R & R & \rho_2 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$ first kind

  $\begin{Bmatrix} \bar{R} & R & \rho_3 \\ R & R & \rho_4 \end{Bmatrix}_{r_1' r_1' r_2' r_4'}$ second kind

  $R = (21;0)$

  $\rho_1, \rho_2, \rho_3 \in R \otimes \bar{R}$

  $\rho_4 \in R \otimes R$

HOMFLYs

Colored knot invariants (HOMFLY) in $[21]$ for non-torus knots

4_1 5_2 6_1 6_2 6_3 7_2 7_3 7_4 7_5 7_6 7_7 8_1 8_2 8_3 8_4 ...

Nontorus knot

4_1 (figure eight)

Pretzel

$$\bar{H}_{\boxplus}(4_1) = \frac{1}{\lambda^3 q^5}\Big(q^5\lambda^6 + (-q^8 - q^6 + q^5 - q^4 - q^2)\lambda^5$$
$$+ (q^{10} - q^9 + 3q^8 - 3q^7 + 5q^6 - 4q^5 + 5q^4 - 3q^3 + 3q^2 - q + 1)\lambda^4$$
$$+ (-2q^{10} + 2q^9 - 5q^8 + 6q^7 - 8q^6 + 7q^5 - 8q^4 + 6q^3 - 5q^2 + 2q - 2)\lambda^3$$
$$+ (q^{10} - q^9 + 3q^8 - 3q^7 + 5q^6 - 4q^5 + 5q^4 - 3q^3 + 3q^2 - q + 1)\lambda^2$$
$$+ (-q^8 - q^6 + q^5 - q^4 - q^2)\lambda + q^5\Big)$$

framing 0.

Symmetries

1. $(q, \lambda) \longmapsto (q^{-1}, \lambda^{-1})$

   Consequence of 4_1 being amphichiral 4_1 6_3 8_3 8_9 8_{12} crossing ≤ 8

2. $(q, \lambda) \longmapsto (q^{-1}, \lambda)$

   Consequence of $\boxplus^T = \widehat{\boxplus}$ $\quad \bar{H}_R[K](q^{-1}, \lambda) = \bar{H}_{\tilde{R}}[K](q, \lambda)$

Topological String

U(N) Chern-Simons theory on S^3 with level k $\quad \longleftrightarrow \quad$ topological B-model

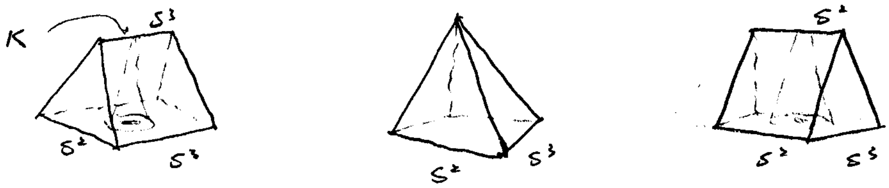
K

$\updownarrow$ $\quad\quad\quad\quad$ $\updownarrow$ mirror symmetry

topological A-model on T^*S^3 w/ N branes on S^3 $\quad \xleftrightarrow{\text{large } N} \quad$ topological A-model on $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$

$g_s = \frac{2\pi}{k+N}$

$t = iNg_s$



Chern-Simons

$$W_R(K) = \langle \mathrm{Tr}_R P \exp \oint_K A \rangle$$

$\downarrow$

F_{cs}

B-model

$W_n^{(g)}(q_1 \cdots q_n)$ $\sim$ instantons ending on Lag. brane in A-model

$\downarrow$

F_{top} brane partition function