High-energy scattering in strongly coupled $\mathcal{N} = 4$ SYM

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Seminar ETH Zürich

based on 1207.4204, 1311.1512, 1405.3658 and 1411.2495
with J. Bartels, J. Kotanski and V. Schomerus

Motivation

- planar $\mathcal{N} = 4$ integrable $\leftrightarrow$ can compute observables for any coupling
- scattering amplitudes particularly interesting
  - functions of kinematical invariants
  - techniques for less symmetric theories
- enormous progress on weak coupling side
- how do amplitudes behave at strong coupling?
  - interpolation to intermediate coupling

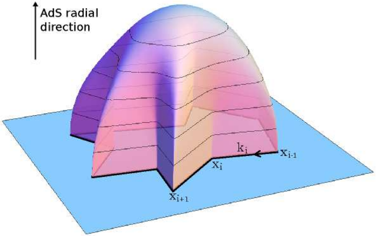
Main results

simple results in high-energy limit at strong coupling:

- MRL corresponds to IR limit of TBA
- $e^{R_6} \sim \left(-(1-u_1)\sqrt{\tilde{u}_2\tilde{u}_3}\right)^{\frac{\sqrt{\lambda}}{2\pi}e_2}$
- 7-point amplitude calculated $\rightarrow$ simple result
- correspondence: Regge cut contributions $\leftrightarrow$ excitations of TBA

Scattering Amplitudes via AdS/CFT

[Alday/Maldacena, A/M/Gaiotto, A/M/Sever/Vieira], [Basso/Sever/Vieira]



[Figure from 1002.2459]

- $\mathrm{A} \sim e^{-\frac{\sqrt{\lambda}}{2\pi}\mathrm{Area}\left(+\frac{\sqrt{\lambda}}{48}\frac{(n-4)(n-5)}{n}\right)}$
- $k_i = x_{i-1} - x_i$
  $\rightarrow$ polygon depends only on $u = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$
- $\mathrm{Y}_{a,s}(\theta)$ generalized cross ratios
  $\rightarrow$ for n gluons: $3n - 15$ Y-functions

$$\boxed{\begin{aligned}\log \mathrm{Y}_{a,s}(\theta) = &- \ m_s \ \cosh\theta \pm \ C_s \\ &+ \sum_{a',s'} \int d\theta' \ \mathcal{K}_{s,s'}^{a,a'} \ \left(\theta - \theta' + \ i\varphi_s - i\varphi_{s'} \ \right) \log\left(1 + \mathrm{Y}_{a',s'}(\theta')\right)\end{aligned}}$$

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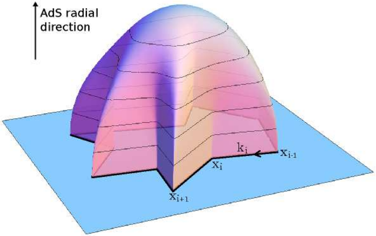

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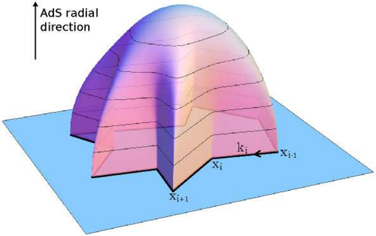

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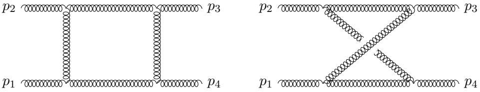
Y-System

$$\boxed{\begin{aligned}\text{Area} &= \text{A}_{\text{div}}\left(x_i\right) + \text{A}_{\text{periods}}\left(m_s,\varphi_s\right) + \Delta(u_i) \\ &\quad + \sum_s \int \frac{d\theta}{2\pi}\left|m_s\right|\cosh\theta\,\log\left[\left(1+\mathrm{Y}_{1,s}\right)\left(1+\mathrm{Y}_{3,s}\right)\left(1+\mathrm{Y}_{2,s}\right)^{\sqrt{2}}\right](\theta)\end{aligned}}$$

- non-divergent piece: remainder function e^R
- Y-system easy to solve numerically
- BUT: not solvable analytically for arbitrary kinematics!

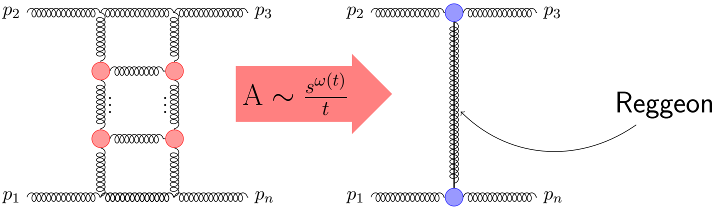
Intermezzo: Regge limit

- consider 1-loop gluon amplitude in Regge limit $s \to \infty$, t fixed

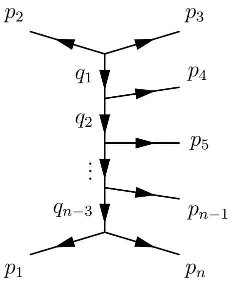


$\xrightarrow{s \to \infty} (\alpha \log s)\, \mathrm{A}_{\mathrm{tree}}$

- leading n-loop contribution $\sim \alpha^n \log^n s$
- in limit $s \to \infty$: $\alpha \log s \sim \mathcal{O}(1)$
  - need to resum leading contributions from all loop orders



Multi-Regge limit



- for $2 \to n-2$ scattering: $3n-10$ Mandelstam invariants
- Multi-Regge limit: rapidities of produced particles strongly ordered
  $\to$ hierarchy for $s_{i\cdots j} := (p_i + \cdots + p_j)^2$

$$\boxed{s \gg s_{3\cdots n-1}, s_{4\cdots n} \gg s_{3\cdots n-2}, \ldots, s_{5\cdots n} \gg \cdots \gg s_{34}, \ldots \gg -t_1, \ldots, -t_{n-3}}$$

- $\mathcal{N}=4$ dual conformal $\to$ choose $3n-15$ cross ratios u_{as}
- kinematical analysis:
  $u_{1s} \to 1,\ u_{2s}, u_{3s} \to 0$ with $\tilde{u}_{2s} = \frac{u_{2s}}{1-u_{1s}} = \mathcal{O}(1), \quad \tilde{u}_{3s} = \frac{u_{3s}}{1-u_{1s}} = \mathcal{O}(1)$

Y-system in the MRL

[1207.4204]

- $u_{as} = \left.\frac{\mathrm{Y}_{2s}}{1+\mathrm{Y}_{2s}}\right|_{\theta=i(k\pi/4-\varphi_s)}$
- demand that cross ratios show behavior predicted by MRL

Example: 6-point case

$$u_2 \to 0 \Rightarrow \mathrm{Y}_2\left(\theta = i\tfrac{\pi}{4}\right) \stackrel{!}{\to} 0$$

$$\log \mathrm{Y}_2\left(\theta = i\frac{\pi}{4}\right) = -\sqrt{2}m\cos\left(\frac{\pi}{4}-\varphi\right) + \sum_{s'}\int d\theta' \mathcal{K}\left(\theta-\theta'\right)\log\underbrace{\left(1+\mathrm{Y}_{s'}\left(\theta'\right)\right)}_{\cong 1+e^{-m_{s'}\cosh\theta'}}$$

Y-system in the MRL

[1207.4204]

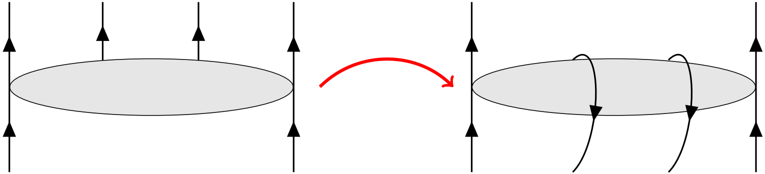
- MRL realized for choice m_s large, $\varphi_s \to -(s-1)\frac{\pi}{4}$, $C_s \to$ const.
- in this limit, integrals in Y-system can be neglected

$$\boxed{\log \mathrm{Y}_{a,s}(\theta) \cong -m_s \cosh\theta \pm C_s + \mathcal{O}(e^{-m})}$$

- $\to$ MRL corresponds to IR limit of TBA
- but: in this limit R trivial
- need to generalize Regge limit $\to$ Regge regions

Multi-Regge regions

- 2^{n-4} regions, corresponding to the signs of E_i
- different regions connected by analytic continuation in $s_i \to s_i e^{i\alpha}$



- for the above example $P_{6,--}$:
  $s_{34} \to e^{i\pi} s_{34} \quad s_{56} \to e^{i\pi} s_{56}, \quad s_{345} \to e^{i\pi} s_{345}, \quad s_{456} \to e^{i\pi} s_{456}$
  $$\Rightarrow u_1 \to e^{-2\pi i} u_1, \quad u_2 \to u_2, \quad u_3 \to u_3$$
- probe analytic structure of amplitude

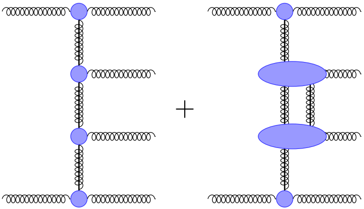
MRL at weak coupling - 6-points

[Bartels/Lipatov/Sabio Vera, Lipatov/Prygarin, Fadin/Lipatov]

- R contains cuts $\Rightarrow$ MRL depends on Regge region:
    - $R_{6,++}$ vanishes in MRL
    - Regge cut appears in Regge region $R_{6,--}$

$$\boxed{e^{R_{6,--}+i\pi\delta}\big|_{\mathrm{MRL}} = \cos\omega_{ab} + i\frac{\lambda}{2}\sum_n \int \frac{d\nu}{\nu^2+\frac{n^2}{4}} |w|^{2i\nu}\, \Phi_{\mathrm{Reg}}(\nu,n) \left(-(1-u_1)\sqrt{\tilde{u}_2\tilde{u}_3}\right)^{-\omega(\nu,n)}}$$

$$e^{R_{6,--}+i\pi\delta}\big|_{\mathrm{MRL}} \quad \text{`='}$$



- universal building blocks: BFKL eigenvalue $\omega(\nu,n)$, impact factor $\Phi_{\mathrm{Reg}}(\nu,n)$

Multi-Regge regions in the Y-system

- continuation in u_{as} $\sim$ continuation in $m, C, \varphi \rightarrow$ [Dorey/Tateo]
- (numerical) inversion of $u_{as} = \frac{\mathrm{Y}_{2s}}{1+\mathrm{Y}_{2s}}\Big|_{\theta=i(k\pi/4-\varphi_s)}$ to find paths for parameters
- solutions of $\mathrm{Y}_{a,s}(\theta) = -1$ can cross real axis

$$\boxed{\begin{aligned}\log \mathrm{Y}_{a,s}(\theta) = &-m_s \cosh\theta \pm C_s \\ &+ \sum_{a',s'} \int d\theta' \mathcal{K}^{a,a'}_{s,s'}(\theta - \theta' + i\varphi_s - i\varphi_{s'}) \log\left(1 + \mathrm{Y}_{a',s'}(\theta')\right)\end{aligned}}$$

- crossing leads to contributions to R

$$\boxed{R' = \int \frac{d\theta}{2\pi} |m_s| \cosh\theta \log\left[(1 + \mathrm{Y}_{a,s}(\theta))\right] \pm i\,|m_s| \sinh(\theta_0) + \ldots}$$

- endpoint explicitly enters R

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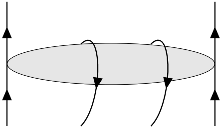
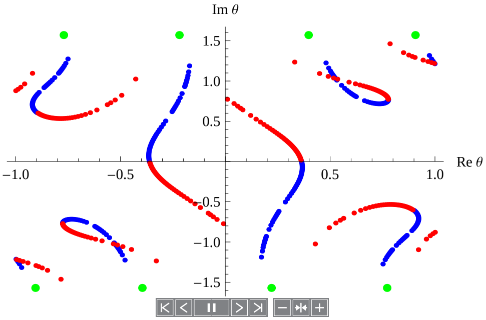
Example: 6-point case

[Bartels/Kotanski/Schomerus, 1311.1512]



$$u_1 \to e^{-2\pi i} u_1,\; u_2 \to u_2,\; u_3 \to u_3$$



Solutions of $\mathrm{Y}_3(\theta) = -1$ along continuation

Determination of endpoints

- endpoints of crossed solutions can be determined analytically

Endpoint condition

$$-1 = \mathrm{Y}_3'(\theta_+) = e^{-m'\cosh(\theta_+)+C'} \cdot \frac{\mathcal{S}_3(\theta_+, \theta_-)}{\mathcal{S}_3(\theta_+, \theta_+)}$$

- for every Regge region get set of BAE which determines remainder function
- numerical input just provides discrete information on crossing

6-point result

- Remainder function has Regge behavior:

$$\boxed{e^{R_{6,--}+i\pi\delta} \sim \left(-(1-u_1)\sqrt{\tilde{u}_2\tilde{u}_3}\right)^{\frac{\sqrt{\lambda}}{2\pi}e_2}}$$

- $e_2 = -\sqrt{2} + \log\left(1+\sqrt{2}\right) \sim -.533 < 0$
- $\delta = \frac{1}{4}\gamma_K \log\sqrt{\tilde{u}_2\tilde{u}_3}$

- weak coupling:

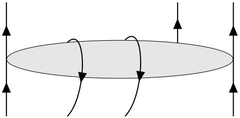
$$e^{R_{6,--}+i\pi\delta}\big|_{\mathrm{MRL}} = i\frac{\lambda}{2}\sum_n \int \frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\,\Phi_{\mathrm{Reg}}(\nu,n)\left(-(1-u_1)\sqrt{\tilde{u}_2\tilde{u}_3}\right)^{-\omega(\nu,n)} + \ldots$$

- dominant saddle point at strong coupling? [Basso/Caron – Huot/Sever]
- cut contribution @ weak coupling $\leftrightarrow$ crossing solution @ strong coupling

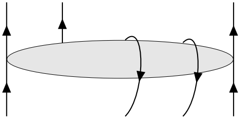
7-point amplitude, predictions from weak coupling

[Bartels/Kormilitzin/Lipatov]

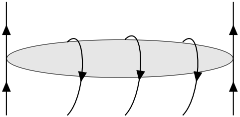
- 7-point cut from two reggeized gluons, as in 6-point case
- predictions obtained from analysis of Regge factorization of BDS ansatz



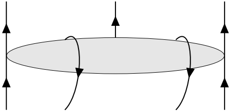
$\rightarrow$ short cut in $1 - u_{11}$



$\rightarrow$ short cut in $1 - u_{12}$

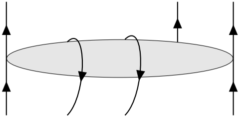


$\rightarrow$ long cut in $(1 - u_{11})(1 - u_{12})$



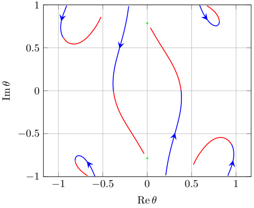
$\rightarrow$ all three cuts contribute

7-point amplitude, paths with short cut



$$u_{11} \to e^{-2i\pi} u_{11},\ u_{21} \to u_{21}, \qquad u_{31} \to u_{31},$$
$$u_{12} \to u_{12}, \qquad u_{22} \to e^{-i\pi} u_{22},\ u_{32} \to e^{i\pi} u_{32}$$

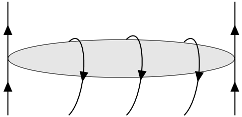
- one pair of crossing solutions
- analogously for mirrored path
- calculation different, we still find:



Crossing solutions of $\mathrm{Y}_{32}(\theta) = -1$

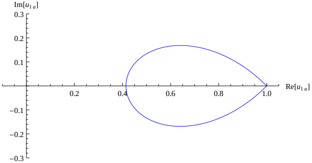
$$\boxed{\begin{aligned} R_{7,--+}(u_{as}) &= R_{6,--}(u_{11}, u_{21}, u_{31}) \\ R_{7,+--}(u_{as}) &= R_{6,--}(u_{12}, u_{22}, u_{32}) \end{aligned}}$$

7-point amplitude, paths with long cut



$$u_{11} \to u_{11},\ u_{21} \to u_{21},\ u_{31} \to u_{31},$$
$$u_{12} \to u_{12},\ u_{22} \to u_{22},\ u_{32} \to u_{32}$$

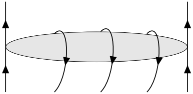
- for 7 points, we get dependent cross ratio $\tilde{u}$
- from kinematics for above path: $\tilde{u} \to e^{-2\pi i}\tilde{u}$
- in conflict with Gram relation: $\tilde{u} \cong \frac{u_{11}+u_{12}-1}{u_{11}u_{12}}$



Deformed path for u_{1a}

- deform path s.th. winding numbers of cross ratios are preserved
- $u_{11} \to e^{2i\pi}\left(1-\sqrt{1-e^{-2i\pi}}\right)u_{11},\ u_{12} \to e^{2i\pi}\left(1-\sqrt{1-e^{-2i\pi}}\right)u_{12}$

7-point amplitude, paths with long cut



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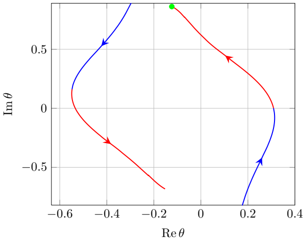
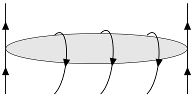
- two pairs of crossing solutions
- both approach same endpoints $\to \theta_\pm = \pm i\frac{\pi}{4}$



Crossing solutions of $\mathrm{Y}_{31}(\theta) = -1$

$$\boxed{R_{7,---}(u_{as}) = R_{6,--}(u_{11}, u_{21}, u_{31}) + R_{6,--}(u_{12}, u_{22}, u_{32})}$$

7-point amplitude, paths with long cut



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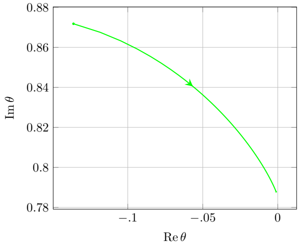
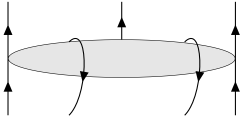


Convergence against endpoint

$$\boxed{R_{7,---}(u_{as}) = R_{6,--}(u_{11}, u_{21}, u_{31}) + R_{6,--}(u_{12}, u_{22}, u_{32})}$$

7-point amplitude, paths with long cut



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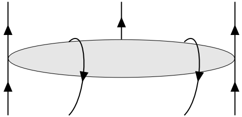
- for this path $\tilde{u} \to e^{-2i\pi} \tilde{u}$, no deformation needed
- no evidence for crossing solutions
  $\Rightarrow R_{7,-+-}(u_{as})$ trivial up to phase
- comparison with weak coupling prediction: possible cancellation?
- path too naive?

7-point amplitude, paths with long cut



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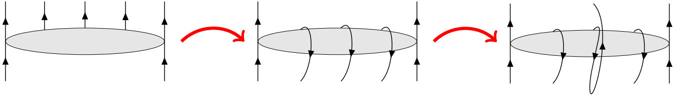

Summary and Outlook

- studied scattering amplitudes in strongly coupled $\mathcal{N} = 4$ SYM
- identified MRL and showed that Y-system equations simplify in MRL
- calculated 6-point and 7-point remainder function
  $\rightarrow$ consistency with weak coupling predictions
- correspondence: Regge cut contributions $\leftrightarrow$ crossing solutions

next steps:

- understand $R_{7,-+-}$: path too naive?
- weak coupling prediction:
  3 Reggeon contribution for 8 points
- understand BAE $\leftrightarrow$ Regge region