

Amplitudes, Form Factors and the Dilatation Operator of $\mathcal{N} = 4$ SYM theory

Matthias Wilhelm, Humboldt University Berlin



ETH Zürich

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- 1 Motivation
- 2 Gauge-invariant local composite operators
- 3 Form factors in the free theory
- 4 One-loop corrections
- 5 Divergences and the dilatation operator
- 6 Two-loop Konishi form factor
- 7 Conclusions and outlook

Motivation to study $\mathcal{N} = 4$ SYM theory

- Gain general understanding of gauge theories
- Develop new techniques for Standard Model calculations
- Strong coupling description via the AdS/CFT correspondence
- Integrability
- “Hydrogen atom of the 21st century”

Motivation to study form factors (1)

$$\begin{aligned} \mathcal{A}(1, \dots, n) \\ = \langle 1, \dots, n | 0 \rangle \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{\mathcal{O}_1 \dots \mathcal{O}_n}(x_1, \dots, x_n) \\ = \langle 0 | \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) | 0 \rangle \end{aligned}$$

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$$\begin{aligned} \mathcal{F}_{\mathcal{O}}(1, \dots, n; x) \\ = \langle 1, \dots, n | \mathcal{O}(x) | 0 \rangle \end{aligned}$$

Picture of a bridge

(Removed in the online version for copyright reasons)

⇒ Form factors as bridge between purely on-shell amplitudes and purely off-shell correlation functions [van Neerven (1986)]
[Boels, Bork, Brandhuber, Engelund, Gehrman, Gurdogan, Henn, Huber, Kazakov, Kniehl, Moch, Mooney, Naculich, Penante, Roiban, Spence, Tarasov, Travaglini, Vartanov, Wen, Yang (2010–2014)]

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Previous studies have focused on $\text{tr}[\phi^{12}\phi^{12}]$ and $\text{tr}[(\phi^{12})^k]$

→ Study form factor of generic operator

[MW(2014)]

Motivation to study form factors (2)

on-shell methods

integrability

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Form factors as bridge between on-shell methods and integrability

→ Revisit spectral problem via on-shell methods

[MW(2014)]

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Gauge invariance

Traces of fields transforming covariantly under the gauge group $SU(N)$:

$$\phi_{AB}, \psi_{ABC\alpha} = \epsilon_{ABCD}\psi_{\alpha}^D, \bar{\psi}_{A\dot{\alpha}}, F_{\mu\nu}, D_{\mu}$$

+ products of such traces

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Planar limit \Rightarrow Sufficient to look at single-trace operators

Pauli matrices $\sigma_{\alpha\dot{\alpha}}^{\mu}$: $\mu, \nu \rightarrow \alpha, \dot{\alpha}$

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Equation of motion, definition of field strength, Bianchi identities

Antisymmetric occurrences of $\alpha, \dot{\alpha}$ at one field

→ Several fields with totally symmetric $\alpha, \dot{\alpha}$

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⇒ Irreducible fields transforming in the
singleton representation \mathcal{V}_S of $\mathfrak{psu}(2, 2|4)$

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Picture of a spin chain

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Oscillator representation of \mathcal{V}_S

Bosonic oscillators: $\mathbf{a}_\alpha, \mathbf{a}^{\dagger\alpha}$ and $\mathbf{b}_{\dot{\alpha}}, \mathbf{b}^{\dagger\dot{\alpha}}$

Fermionic oscillators: $\mathbf{d}_A, \mathbf{d}^{\dagger A}$

$$[\mathbf{a}_\alpha, \mathbf{a}^{\dagger\beta}] = \delta_\alpha^\beta, \quad [\mathbf{b}_{\dot{\alpha}}, \mathbf{b}^{\dagger\dot{\beta}}] = \delta_{\dot{\alpha}}^{\dot{\beta}}, \quad \{\mathbf{d}_A, \mathbf{d}^{\dagger B}\} = \delta_A^B$$

Oscillator picture

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Irreducible fields

$$D^k F \quad \cong \quad (\mathbf{a}^\dagger)^{k+2} (\mathbf{b}^\dagger)^k \quad \mathbf{d}^{\dagger 1} \mathbf{d}^{\dagger 2} \mathbf{d}^{\dagger 3} \mathbf{d}^{\dagger 4} |0\rangle$$

$$D^k \psi_{ABC} \quad \cong \quad (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \quad \mathbf{d}^{\dagger A} \mathbf{d}^{\dagger B} \mathbf{d}^{\dagger C} |0\rangle$$

$$D^k \phi_{AB} \quad \cong \quad (\mathbf{a}^\dagger)^k \quad (\mathbf{b}^\dagger)^k \quad \mathbf{d}^{\dagger A} \mathbf{d}^{\dagger B} |0\rangle$$

$$D^k \bar{\psi}_A \quad \cong \quad (\mathbf{a}^\dagger)^k \quad (\mathbf{b}^\dagger)^{k+1} \mathbf{d}^{\dagger A} |0\rangle$$

$$D^k \bar{F} \quad \cong \quad (\mathbf{a}^\dagger)^k \quad (\mathbf{b}^\dagger)^{k+2} |0\rangle$$

Spin chain, dilatation operator and integrability

Dilatation operator measures (anomalous) scaling dimensions

→ Observables in a CFT

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$\mathfrak{su}(2)$ sector: single-trace operators built from $\uparrow = \phi_{24}$ and $\downarrow = \phi_{34}$

Heisenberg XXX spin chain $(\mathfrak{D}_2)_{ii+1} = 2(\mathbb{1} - \mathbb{P})_{ii+1}$

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Example

$$\mathfrak{D}_2 \text{tr}[\uparrow\uparrow\downarrow\downarrow] = 4 \text{tr}[\uparrow\uparrow\downarrow\downarrow] - 4 \text{tr}[\uparrow\downarrow\uparrow\downarrow]$$

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Spectral problem can be solved by Bethe ansatz techniques

[Minahan, Zarembo (2002)] [Beisert (2003)] [Beisert, Staudacher (2003)]

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Fourier transform to momentum space

$$\begin{aligned}\mathcal{F}_{\mathcal{O}}(1, \dots, n; q) &= \int d^4x e^{-iqx} \langle 1, \dots, n | \mathcal{O}(x) | 0 \rangle \\ &= \delta^4 \left(q - \sum_{i=1}^n p_i \right) \langle 1, \dots, n | \mathcal{O}(0) | 0 \rangle\end{aligned}$$

Super spinor helicity variables for super form factors

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Nair's $\mathcal{N} = 4$ on-shell super field

$$\Phi = g^+ + \eta^A \bar{\psi}_A + \frac{1}{2!} \eta^A \eta^B \phi_{AB} + \eta^A \eta^B \eta^C \psi_{ABC} + \eta^1 \eta^2 \eta^3 \eta^4 g^-$$

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Colour-ordered super form factors

$$\mathcal{F}_{\mathcal{O}}(1, \dots, n; q) = \sum_{\sigma \in \mathbb{S}_n / \mathbb{Z}_n} \text{tr}[T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}] \hat{\mathcal{F}}_{\mathcal{O}}(\sigma(1), \dots, \sigma(n); q) + \text{multi-trace terms}$$

Computing form factors

Form factors can be obtained by

- BCFW and MHV recursion relations
- (generalised) unitarity
- symbols
- colour kinematic duality
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MHV amplitude

[Parke, Taylor (1986)]

$$\hat{A}^{(0),\text{MHV}}(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = \frac{\langle ij \rangle^4 \delta^4(\sum_{k=1}^n p_k)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

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MHV form factor

[Brandhuber, Spence, Travaglini, Yang (2011)]

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Computation via Feynman rules in the free theory

$$D_{\alpha\dot{\alpha}} \quad : \quad \rightarrow \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$$

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Form factors as spin chains

Colour-ordered minimal super form factor for generic operator \mathcal{O}

$$\hat{\mathcal{F}}_{\mathcal{O}}(\Lambda_1, \dots, \Lambda_L; q) = L\delta^4 \left(q - \sum_{i=1}^L \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \right) \left(\begin{array}{l} \mathbf{a}_i^{\dagger\alpha} \rightarrow \lambda_i^\alpha \\ \mathbf{b}_i^{\dagger\dot{\alpha}} \rightarrow \tilde{\lambda}_i^{\dot{\alpha}} \\ \mathbf{d}_i^{\dagger A} \rightarrow \eta_i^A \\ \text{in oscillator picture} \end{array} \right)$$

with $\Lambda_i = (\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$

Minimal tree-level form factor = form factor in free theory

[MW(2014)]

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Minimal tree-level form factor = form factor in free theory

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⇒ Form factors exactly implement the spin chain of $\mathcal{N} = 4$ SYM theory in the language of scattering amplitudes

[MW(2014)]

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General ansatz from integral basis

$$\begin{aligned}
 q \Rightarrow \text{circle } \hat{F}_O \Rightarrow p_3 &= \sum_{i,j,k,l} c_{\text{box}}^{(i,j,k,l)} \text{box diagram} + \sum_{i,j,k} c_{\text{triangle}}^{(i,j,k)} \text{triangle diagram} \\
 &+ \sum_{i,j} c_{\text{bubble}}^{(i,j)} \text{bubble diagram} + \text{rational terms}
 \end{aligned}$$

General ansatz from integral basis

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⇒ Sufficient to determine coefficients

General ansatz from integral basis

$$\begin{aligned}
 q \Rightarrow \hat{\mathcal{F}}_O \Rightarrow p_3 &= \sum_{i,j,k,l} c_{\text{box}}^{(i,j,k,l)} \text{ [box diagram]} + \sum_{i,j,k} c_{\text{triangle}}^{(i,j,k)} \text{ [triangle diagram]} \\
 &+ \sum_{i,j} c_{\text{bubble}}^{(i,j)} \text{ [bubble diagram]} + \text{rational terms}
 \end{aligned}$$

⇒ Sufficient to determine coefficients

→ Further simplifications for minimal form factors

Ansatz for minimal form factor

Simplified ansatz:

$$q \rightarrow \hat{\mathcal{F}}_0 \rightarrow p_3 = \sum_i c_{\text{triangle}}^{i,i+1} \text{triangle diagram} + \sum_i c_{\text{bubble}}^{i,i+1} \text{bubble diagram} + \text{rational terms}$$

⇒ Determine coefficients via cuts

Ansatz for minimal form factor

Simplified ansatz:

The diagrammatic equation shows the decomposition of a form factor into a sum of triangle and bubble diagrams. On the left, a central circle labeled $\hat{\mathcal{F}}_{\mathcal{O}}$ has an incoming arrow from the left labeled q and several outgoing arrows labeled $p_1, p_2, p_3, \dots, p_L$. This is equated to a sum over i of coefficients $c_{\text{triangle}}^{i,i+1}$ multiplied by a triangle diagram. The triangle diagram has an incoming arrow q on the left, and outgoing arrows p_{i-1} (top), p_i (right), and p_{i+1} (bottom). Below this is the text "+ rational terms". To the right, there is a plus sign followed by a sum over i of coefficients $c_{\text{bubble}}^{i,i+1}$ multiplied by a bubble diagram. The bubble diagram has an incoming arrow q on the left, and outgoing arrows p_{i-1} (top), p_i (right), and p_{i+1} (bottom).

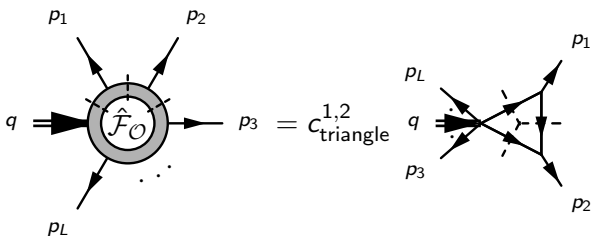
$$\hat{\mathcal{F}}_{\mathcal{O}}(q, p_1, p_2, \dots, p_L) = \sum_i c_{\text{triangle}}^{i,i+1} \text{triangle}(q, p_{i-1}, p_i, p_{i+1}) + \text{rational terms} + \sum_i c_{\text{bubble}}^{i,i+1} \text{bubble}(q, p_{i-1}, p_i, p_{i+1})$$

⇒ Determine coefficients via cuts

Cut: $\frac{1}{l^2} \rightarrow \delta(l^2)\Theta(l_0)$

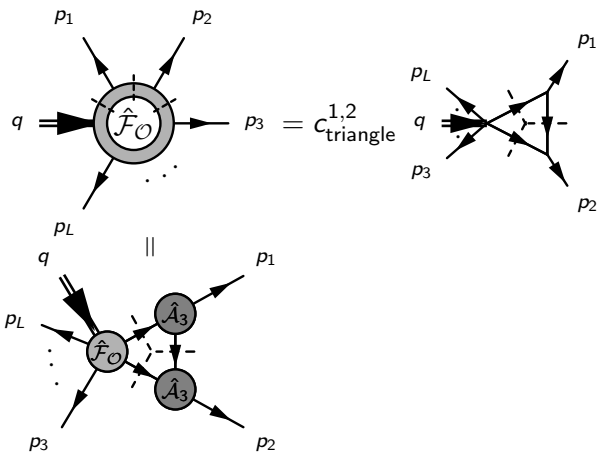
Triple cut and triangle coefficient

Triple cut between p_1 , p_2 and the rest of the diagram:

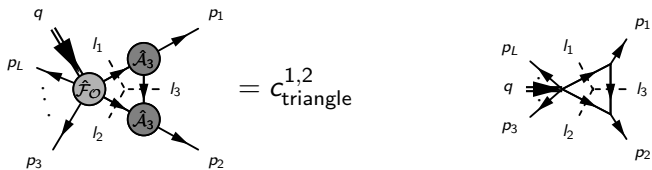


Triple cut and triangle coefficient

Triple cut between p_1 , p_2 and the rest of the diagram:



Triple cut and triangle coefficient



Triple cut and triangle coefficient

Integration over all intermediate degrees of freedom:

$$\int \prod_{i=1}^3 d\Lambda_{l_i} \text{ [Diagram 1] } = c_{\text{triangle}}^{1,2} \int \prod_{i=1}^3 d\Lambda_{l_i} \text{ [Diagram 2] }$$

Diagram 1: A central vertex $\hat{\mathcal{F}}_0$ is connected to three external legs l_1, l_2, l_3 and three internal legs l_1, l_2, l_3 . The internal legs are connected to two vertices $\hat{\mathcal{A}}_3$. External momenta are q, p_L, p_1, p_2, p_3 .

Diagram 2: A triangle diagram with vertices $\hat{\mathcal{A}}_3$ and external legs l_1, l_2, l_3 . External momenta are q, p_L, p_1, p_2, p_3 .

with $d\Lambda_{l_i} = d^2\lambda_{l_i} d^2\tilde{\lambda}_{l_i} d^4\eta_{l_i}$

Triple cut and triangle coefficient

The triangle coefficient:

$$C_{\text{triangle}}^{1,2} = \frac{\int \prod_{i=1}^3 d\Lambda_{l_i} \text{ (Diagram 1)}}{\int \prod_{i=1}^3 d\Lambda_{l_i} \text{ (Diagram 2)}}$$

The diagram in the numerator (Diagram 1) shows a central vertex \hat{f}_O with external momenta q , p_L , and p_3 . It is connected to two vertices \hat{A}_3 via internal lines l_1 and l_2 . The vertices \hat{A}_3 are further connected to external momenta p_1 and p_2 via lines l_3 . Dashed lines indicate the triple cut.

The diagram in the denominator (Diagram 2) shows the same external momenta and internal lines, but with a different internal structure where the lines l_1 and l_2 cross, representing a different cut configuration.

Triple cut and triangle coefficient

The triangle coefficient:

$$\begin{aligned}
 C_{\text{triangle}}^{1,2} &= \frac{\int \prod_{i=1}^3 d\Lambda_{l_i} \text{ (Diagram 1)}}{\int \prod_{i=1}^3 d\Lambda_{l_i} \text{ (Diagram 2)}} \\
 &= -(p_1 + p_2)^2 \hat{\mathcal{F}}_{\mathcal{O}}^{(0)}(\Lambda_1, \Lambda_2, \Lambda_3, \dots, \Lambda_L; q)
 \end{aligned}$$

Diagram 1 (Numerator): A central vertex $\hat{\mathcal{F}}_{\mathcal{O}}$ is connected to three internal lines l_1, l_2, l_3 . Line l_1 connects to a vertex $\hat{\mathcal{A}}_3$, which is connected to line l_3 . Line l_2 connects to another vertex $\hat{\mathcal{A}}_3$, which is connected to line l_3 . External momenta are q (incoming), p_L (incoming), p_3 (incoming), p_1 (outgoing), and p_2 (outgoing).

Diagram 2 (Denominator): A central vertex $\hat{\mathcal{F}}_{\mathcal{O}}$ is connected to three internal lines l_1, l_2, l_3 . Line l_1 connects to a vertex $\hat{\mathcal{A}}_3$, which is connected to line l_3 . Line l_2 connects to another vertex $\hat{\mathcal{A}}_3$, which is connected to line l_3 . External momenta are q (incoming), p_L (incoming), p_3 (incoming), p_1 (outgoing), and p_2 (outgoing).

Triple cut and triangle coefficient

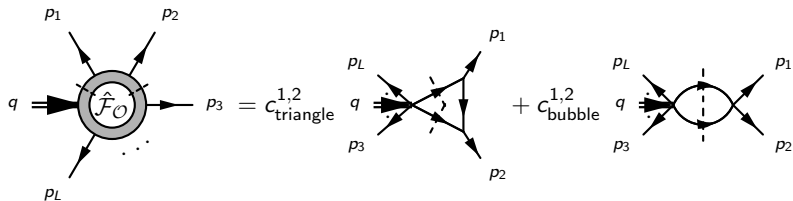
The triangle coefficient:

$$c_{\text{triangle}}^{1,2} = \frac{\int \prod_{i=1}^3 d\Lambda_{l_i} \text{ (Diagram 1)}}{\int \prod_{i=1}^3 d\Lambda_{l_i} \text{ (Diagram 2)}}$$
$$= -(p_1 + p_2)^2 \hat{\mathcal{F}}_{\mathcal{O}}^{(0)}(\Lambda_1, \Lambda_2, \Lambda_3, \dots, \Lambda_L; q)$$

⇒ Universal for all operators

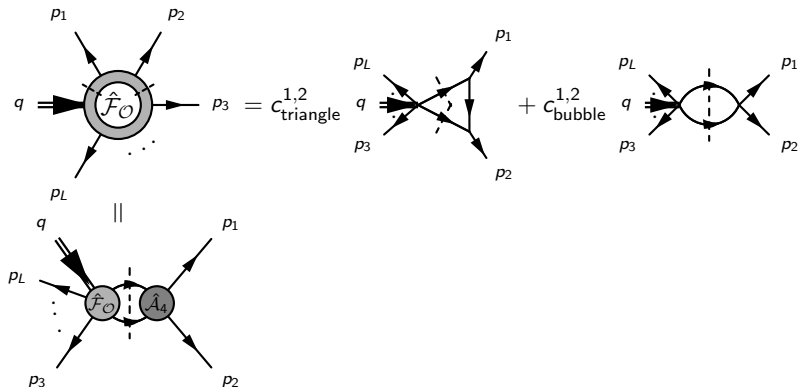
Double cut and bubble coefficient

Double cut between p_1, p_2 and the rest of the diagram:



Double cut and bubble coefficient

Double cut between p_1, p_2 and the rest of the diagram:



Double cut and bubble coefficient

The bubble coefficient:

$$C_{\text{bubble}}^{1,2} = \frac{\int \prod_{i=1}^2 d\Lambda_{l_i} \left(\begin{array}{c} q \\ p_L \\ \vdots \\ p_3 \end{array} \begin{array}{c} \hat{\mathcal{F}}_O \\ \hat{\mathcal{A}}_4 \end{array} \begin{array}{c} l_1 \\ l_2 \\ p_1 \\ p_2 \end{array} - C_{\text{triangle}}^{1,2} \begin{array}{c} p_L \\ q \\ p_3 \end{array} \begin{array}{c} l_1 \\ l_2 \\ p_1 \\ p_2 \end{array} \right)}{\int \prod_{i=1}^2 d\Lambda_{l_i} \begin{array}{c} p_L \\ q \\ p_3 \end{array} \begin{array}{c} l_2 \\ l_1 \\ p_1 \\ p_2 \end{array}}$$

Double cut and bubble coefficient

The bubble coefficient:

$$C_{\text{bubble}}^{1,2} = \frac{\int \prod_{i=1}^2 d\Lambda_{l_i} \left(\begin{array}{c} q \\ p_L \\ \vdots \\ p_3 \end{array} \begin{array}{c} \hat{\mathcal{F}}_O \\ \hat{\mathcal{A}}_4 \end{array} \begin{array}{c} l_1 \\ l_2 \\ p_1 \\ p_2 \end{array} - C_{\text{triangle}}^{1,2} \begin{array}{c} p_L \\ q \\ p_3 \end{array} \begin{array}{c} l_1 \\ l_2 \end{array} \begin{array}{c} p_1 \\ p_2 \end{array} \right)}{\int \prod_{i=1}^2 d\Lambda_{l_i} \begin{array}{c} p_L \\ q \\ p_3 \end{array} \begin{array}{c} l_2 \\ l_1 \end{array} \begin{array}{c} p_1 \\ p_2 \end{array}}$$

$$= B_{12} \hat{\mathcal{F}}_O^{(0)}(\Lambda_1, \Lambda_2, \Lambda_3, \dots, \Lambda_L; q)$$

$$B_{ii+1} \hat{\mathcal{F}}_{\mathcal{O}}^{(0)}(\Lambda_1, \dots, \Lambda_L; q) =$$
$$-2\delta_{C_i,0} \int_0^{\pi/2} d\theta \cot \theta \left(\hat{\mathcal{F}}_{\mathcal{O}}^{(0)}(\Lambda_1, \dots, \Lambda_i, \Lambda_{i+1}, \dots, \Lambda_L; q) \right. \\ \left. - \hat{\mathcal{F}}_{\mathcal{O}}^{(0)}(\Lambda_1, \dots, \Lambda'_i, \Lambda'_{i+1}, \dots, \Lambda_L; q) \right)$$

with

$$\begin{pmatrix} \Lambda'_i \\ \Lambda'_{i+1} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \Lambda_i \\ \Lambda_{i+1} \end{pmatrix}, \quad \Lambda_i = (\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$$

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Polynomial in $\cos \theta$ and $\sin \theta$

⇒ Evaluates to Euler β -function or harmonic number

Bubble coefficient operator

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with

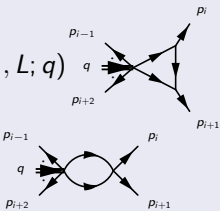
$$\begin{pmatrix} \Lambda'_i \\ \Lambda'_{i+1} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \Lambda_i \\ \Lambda_{i+1} \end{pmatrix}, \quad \Lambda_i = (\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$$

Polynomial in $\cos \theta$ and $\sin \theta$

⇒ Evaluates to Euler β -function or harmonic number

Operator \mathcal{O} in $\mathfrak{su}(2)$ sector ⇒ $B_{ii+1} = -(\mathbb{1} - \mathbb{P})_{ii+1}$

One-loop minimal form factor of a generic operator \mathcal{O}

$$\begin{aligned}
 \hat{\mathcal{F}}_{\mathcal{O}}^{(1)}(1, \dots, L; q) = & - \sum_{i=1}^L (p_i + p_{i+1})^2 \hat{\mathcal{F}}_{\mathcal{O}}^{(0)}(1, \dots, L; q) \quad q \\
 & + \sum_{i=1}^L B_{ii+1} \hat{\mathcal{F}}_{\mathcal{O}}^{(0)}(1, \dots, L; q) \quad q \\
 & + \text{rational terms}
 \end{aligned}$$


[MW(2014)]

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IR divergence from triangle integral (universal)

$$\frac{\hat{\mathcal{F}}_{\mathcal{O}}^{(1)}(1, \dots, L; q)}{\hat{\mathcal{F}}_{\mathcal{O}}^{(0)}(1, \dots, L; q)} \Big|_{\text{IR}} = -\frac{1}{\varepsilon^2} \sum_{i=1}^L (-s_{i i+1})^{-\varepsilon}, \quad D = 4 - 2\varepsilon$$

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Agrees with universal BDS-ansatz-type form

[Bern, Dixon, Smirnov (2005)]

$$\begin{aligned} & \log \left(\frac{\hat{\mathcal{F}}_{\mathcal{O}}(1, \dots, n; q)}{\hat{\mathcal{F}}_{\mathcal{O}}^{(0)}(1, \dots, n; q)} \right) \\ &= \sum_{l=1}^{\infty} g^{2l} \left[-\frac{\gamma_{\text{cusp}}^{(l)}}{8(l\varepsilon)^2} - \frac{\mathcal{G}_0^{(l)}}{4l\varepsilon} \right] \sum_{i=1}^n (-s_{i i+1})^{-l\varepsilon} + \text{Fin}(g^2) + \mathcal{O}(\varepsilon) \end{aligned}$$

(Form checked for BPS form factors in [Brandhuber, Spence, Travaglini, Yang (2010)], [Gehrmann, Henn, Huber (2011)], [Brandhuber, Travaglini, Yang (2012)], [Brandhuber, Penante, Travaglini, Wen (2014)])

Operator renormalisation

$$\mathcal{O}_{\text{ren}}^a = Z^a_b \mathcal{O}_{\text{bare}}^b, \quad Z^a_b = \delta^a_b + g^2 (\mathcal{Z}^{(1)})^a_b + \mathcal{O}(g^3)$$

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UV-finiteness of renormalised form factor \Rightarrow

$$\begin{aligned} (\mathcal{Z}^{(1)})^a_b \hat{\mathcal{F}}_{\mathcal{O}_{\text{bare}}^b}^{(0)}(1, \dots, L; q) &= - \hat{\mathcal{F}}_{\mathcal{O}_{\text{bare}}^a}^{(1)}(1, \dots, L; q) \Big|_{\text{UV}} \\ &= - \left(\sum_{i=1}^L B_{i,i+1} \right)^a_b \hat{\mathcal{F}}_{\mathcal{O}_{\text{bare}}^b}^{(0)}(1, \dots, L; q) \frac{1}{\epsilon} \end{aligned}$$

Anomalous part of dilatation operator

$$\delta\mathcal{D} = \lim_{\varepsilon \rightarrow 0} \varepsilon g \frac{\partial}{\partial g} \ln \mathcal{Z}$$

Dilatation operator

Anomalous part of dilatation operator

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Complete one-loop dilatation operator

$$\mathcal{D}_2 = \frac{1}{g^2} \lim_{\varepsilon \rightarrow 0} \varepsilon g \frac{\partial}{\partial g} \left(g^2 \mathcal{Z}^{(1)} \right) = -2 \sum_{i=1}^L B_{i i+1}$$

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⇒ Agrees with result of [Beisert (2003)] in formulation of [Zwiebel (2007)] after replacing oscillators by super spinor helicity variables. (Proof of a connection between amplitudes and dilatation operator which was observed in [Zwiebel (2011)].)

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- Prime example of non-protected operators
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- Prime example of non-protected operators

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- Eigenstate under renormalisation:

$$\mathcal{K}_{\text{ren}} = \mathcal{Z} \mathcal{K}_{\text{bare}} \quad \text{with} \quad \mathcal{Z} = \exp \left[\sum_{\ell=1}^{\infty} \frac{g^{2\ell} \gamma_{\mathcal{K}}^{(\ell)}}{2\ell\epsilon} \right]$$

- Anomalous dimension $\gamma_{\mathcal{K}}$ known via field theory up to five loops [Eden, Heslop, Korchemsky, Smirnov, Sokatchev (2012)] and via integrability up to ten loops [Marboe, Volin (2014)]

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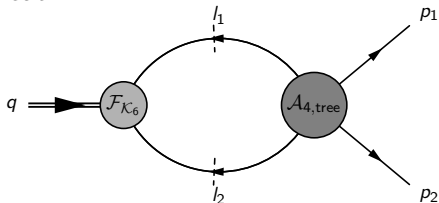
- Using the on-shell super field

$$\Rightarrow \mathcal{K}_6 = \frac{1}{8} \epsilon^{ABCD} \text{tr}[\phi_{AB} \phi_{CD}]$$

- $\mathcal{K}_6 \neq \mathcal{K}$ unless $D = 4$

One-loop Konishi form factor

Planar double cut

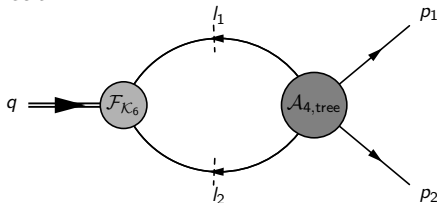


$$\begin{aligned}
 f_{\mathcal{K}_6,(\phi,\phi)}^{(1)} \Big|_{q^2} &= \frac{F_{\mathcal{K}_6,(\phi,\phi)}^{(1)}}{F_{\mathcal{K}_6,(\phi,\phi)}^{(0)}} \Big|_{q^2} = \dots = \int d\text{PS}_{l_1, l_2} \left(\frac{\langle l_1 l_2 \rangle \langle 12 \rangle}{\langle 1 l_1 \rangle \langle 2 l_2 \rangle} + 6 \frac{\langle l_1 2 \rangle \langle l_2 1 \rangle}{\langle 12 \rangle \langle l_1 l_2 \rangle} \right) \\
 &= -s_{12} \text{ (triangle diagram) } + 6 \frac{(l_1 + p_2)^2}{s_{12}} \text{ (bubble diagram) }
 \end{aligned}$$

[Nandan, Sieg, MW, Yang (2014)]

One-loop Konishi form factor

Planar double cut



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 f_{\mathcal{K}_6,(\phi,\phi)}^{(1)} \Big|_{q^2} &= \frac{F_{\mathcal{K}_6,(\phi,\phi)}^{(1)}}{F_{\mathcal{K}_6,(\phi,\phi)}^{(0)}} \Big|_{q^2} = \dots = \int d\text{PS}_{l_1,l_2} \left(\frac{\langle l_1 l_2 \rangle \langle 12 \rangle}{\langle 1 l_1 \rangle \langle 2 l_2 \rangle} + 6 \frac{\langle l_1 2 \rangle \langle l_2 1 \rangle}{\langle 12 \rangle \langle l_1 l_2 \rangle} \right) \\
 &= -s_{12} \text{ (triangle diagram)} + 6 \frac{(l_1 + p_2)^2}{s_{12}} \text{ (bubble diagram)}
 \end{aligned}$$

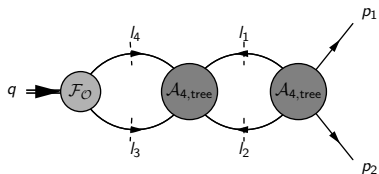
Lift and Passarino-Veltman reduction

$$f_{\mathcal{K}_6,(\phi,\phi)}^{(1)} = 2 \left(-s_{12} \text{ (triangle diagram)} - 3 \text{ (bubble diagram)} \right)$$

[Nandan, Sieg, MW, Yang (2014)]

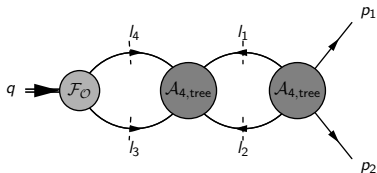
Two-point Konishi form factor

Planar double-double cut

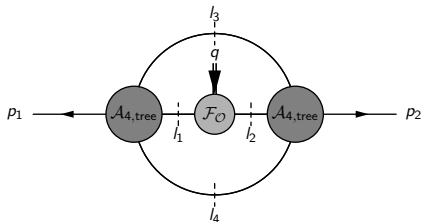


Two-point Konishi form factor

Planar double-double cut

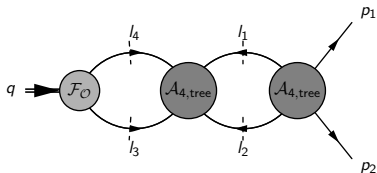


Non-planar double-double cut

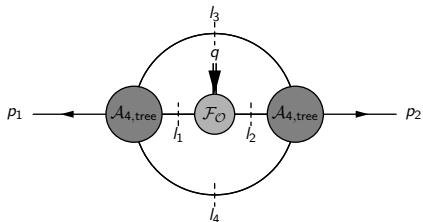


Two-point Konishi form factor

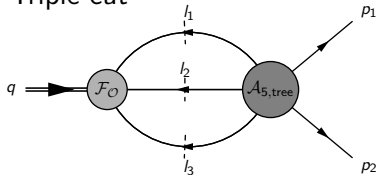
Planar double-double cut



Non-planar double-double cut



Triple cut



Two-point Konishi form factor

Final result:

$$f_{\mathcal{K}_{6,2}}^{(2)} = s_{12}^2 \left(4 \underbrace{\left(\text{triangle} + \text{cross} \right)}_{f_{\text{BPS},2}^{(2)}} + \text{triangle} + \text{cross} \right) - 6(l+p_1)^2(l+p_2)^2 \left(4 \underbrace{\left(\text{triangle} + \text{cross} \right)}_{f_{\text{BPS},2}^{(2)}} + \text{triangle} + \text{cross} \right) + \frac{36}{s_{12}} \text{circle}$$

The equation consists of three terms. The first term is s_{12}^2 multiplied by a sum of two diagrams: a triangle with two vertical lines and a crossed triangle. This sum is labeled $f_{\text{BPS},2}^{(2)}$ with a brace underneath. The second term is $-6(l+p_1)^2(l+p_2)^2$ multiplied by a sum of two diagrams: a triangle with an arrow on the top edge and two vertical lines, and a crossed triangle with an arrow on the top edge. The third term is $+\frac{36}{s_{12}}$ multiplied by a circle diagram with two external lines.

[Nandan, Sieg, MW, Yang (2014)]

Subtleties (1)

Divergences \Rightarrow Regularise in $D = 4 - 2\epsilon$ dimensions

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Divergences \Rightarrow Regularise in $D = 4 - 2\epsilon$ dimensions

$\mathcal{N} = 4$ SYM theory

Dimensional reduction of $\mathcal{N} = 1$ SYM theory in $D = 10$ to $D = 4$

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Divergences \Rightarrow Regularise in $D = 4 - 2\epsilon$ dimensions

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Gauge field A_M , $M = 1, \dots, 10$

$$\begin{aligned} \rightarrow & A_\mu, \quad \mu = 1, \dots, D = 4 - 2\epsilon \\ & \phi_I, \quad I = 1, \dots, 10 - D = 6 + 2\epsilon \end{aligned}$$

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Metric $\eta_{MN} = \eta_{\mu\nu} - \delta_{IJ}$

Feynman diagrams

Vector index loop: $\eta_{\mu\nu}\eta^{\mu\nu} = D = 4 - 2\epsilon$

Scalar index loop: $\delta_{IJ}\delta^{IJ} = 10 - D = 6 + 2\epsilon$

Subtleties (2)

Feynman diagrams

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Only interaction vertices in index loop:

Dimensional reduction

→ gluon + scalar

→ $\eta_{\mu\nu}\eta^{\mu\nu} + \delta_{IJ}\delta^{IJ} = 10$

⇒ independent of ε



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Also operator in index loop:

Dependent of ε !

E.g. $\mathcal{K} = \delta^{IJ} \text{tr}[\phi_I \phi_J]$

Subtleties (3)

Generic multi-loop diagram with incoming operator $\mathcal{O} = \text{tr}[\phi_I \phi_J]$ and outgoing scalar fields ϕ_K and ϕ_L . R-charge conservation:

The diagram shows an equality between a multi-loop diagram and a sum of three diagrams. On the left, a grey circle with four white dots is connected to two external lines labeled p_1 and p_2 . This is equal to the sum of three diagrams, each with a grey circle and two external lines labeled I, J and K, L . Diagram (a) has two blue arcs, (b) has two green arcs, and (c) has two red arcs.

(a) $\delta_{IK} \delta_{JL}$ (b) $\delta_{IL} \delta_{JK}$ (c) $\delta_{IJ} \delta_{KL}$

[Nandan, Sieg, MW, Yang (2014)]

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$$q = \text{Diagram} = \underbrace{\text{(a)} \delta_{IK} \delta_{JL} + \text{(b)} \delta_{IL} \delta_{JK}}_{f_{\text{BPS}}, \mathcal{O}_{\text{BPS}} = \text{tr}[\phi_I \phi_J]} + \text{(c)} \delta_{IJ} \delta_{KL}$$

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Generic multi-loop diagram with incoming operator $\mathcal{O} = \text{tr}[\phi_I \phi_J]$ and outgoing scalar fields ϕ_K and ϕ_L . R-charge conservation:

$$\begin{aligned}
 q &= \text{Diagram} = \text{(a)} + \text{(b)} + \text{(c)} \\
 &= \underbrace{\delta_{IK} \delta_{JL}}_{\text{(a)}} + \underbrace{\delta_{IL} \delta_{JK}}_{\text{(b)}} + \underbrace{\delta_{IJ} \delta_{KL}}_{\text{(c)}} \\
 &= \underbrace{f_{\text{BPS}}, \mathcal{O}_{\text{BPS}} = \text{tr}[\phi_I \phi_J]}_{\text{BPS part}} + \underbrace{f_{\mathcal{K}}, \mathcal{K} = \delta^{IJ} \text{tr}[\phi_I \phi_J]}_{\text{non-BPS part}}
 \end{aligned}$$

[Nandan, Sieg, MW, Yang (2014)]

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Generic multi-loop diagram with incoming operator $\mathcal{O} = \text{tr}[\phi_I \phi_J]$ and outgoing scalar fields ϕ_K and ϕ_L . R-charge conservation:

$$q = \text{Diagram with four internal lines} = \text{(a)} + \text{(b)} + \text{(c)}$$

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 \quad
 $\underbrace{\text{(b)} \delta_{IL} \delta_{JK}}_{f_{\text{BPS}}, \mathcal{O}_{\text{BPS}} = \text{tr}[\phi_I \phi_J]}$
 \quad
 $\text{(c)} \delta_{IJ} \delta_{KL}$

$\underbrace{\text{(a)} \text{ (b)} \text{ (c)}}_{f_{\mathcal{K}}, \mathcal{K} = \delta^{IJ} \text{tr}[\phi_I \phi_J]}$

(a), (b) No scalar index loop involving \mathcal{O}

[Nandan, Sieg, MW, Yang (2014)]

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(a), (b) No scalar index loop involving \mathcal{O}

(c) **One scalar index loop** involving \mathcal{O}

→ Should have $\delta^{IJ} \delta_{IJ} = N_\phi = 6 + 2\epsilon$ instead of $N_\phi = 6$

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Lift intermediate states:

$$f_{\mathcal{K}_{6,n}}^{(\ell)} = f_{\text{BPS},n}^{(\ell)} + \tilde{f}_{\mathcal{K}_{6,n}}^{(\ell)}$$

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⇒ New kind of rational term at $\ell = 1$, $\frac{1}{\epsilon}$ -pole altered at $\ell = 2$

[Nandan, Sieg, MW, Yang (2014)]

Universal IR structure

$$\begin{aligned}(\log f_{\mathcal{K},\text{ren}})^{(2)} &= \left(f_{\mathcal{K},\text{bare}}^{(2)} + \mathcal{Z}_{\mathcal{K}}^{(1)} f_{\mathcal{K},\text{bare}}^{(1)} + \mathcal{Z}_{\mathcal{K}}^{(2)} \right) \\ &\quad - \frac{1}{2} \left(f_{\mathcal{K},\text{bare}}^{(1)} + \mathcal{Z}_{\mathcal{K}}^{(1)} \right)^2\end{aligned}$$

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Agrees with results of [Anselmi, Grisar, Johansen (1997)], [Anselmi, Freedman, Grisar, Johansen (1997)] and [Bianchi, Kovacs, Rossi, Stanev (1999,2000)], [Eden, Schubert, Sokatchev (2000)]

[Nandan, Sieg, MW, Yang (2014)]

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[MW (2014)]

Conclusions

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[Nandan, Sieg, MW, Yang (2014)]

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