Classical and quantum integrability of strings on $AdS_3 \times S^3 \times T^4$ with mixed flux

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Outline

AdS$_3$ integrability with mixed flux

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AdS_3 integrability: brief reminder

- $AdS_3 \times S^3 \times T^4$ background preserves 16 supersymmetries.
- IIB supergravity backgroud is near horizon limit of mixed NS5-NS1 + D5-D1 solutions.
- $R_{AdS_3} = R_{S^3}$.
- S-duality transforms NS-NS 3-form flux into RR such that coefficients χ, κ are related $\chi^2 + \kappa^2 = 1$
- Pure NS-NS limit is given by superstring generalization of $SL(2) \times SU(2)$ WZW model and spectrum can be found in RNS formalation (chiral decomposition of WZW)
- Pure RR limit can be described in GS formulation giving the supercoset $\dfrac{PSU(1,1|2) \times PSU(1,1|2)}{SL(2) \times SU(2)} \times U(1)^4$ and is classically integrable due to Z_4 symmetry

[A.B.,B.Stefanski,K.Zarembo, 2009]

AdS_3 integrability: brief reminder

The mixed flux theory was shown classically integrable in all the range $0 \leqslant \chi \leqslant 1$ [A.Cagnazzo, K.Zarembo, 2012]

Quantum S-matrix with mixed flux has the same χ - independent symmetry as in pure RR limit, but its representation is χ - dependent: χ enters through dispersion relation

There are novel features of AdS_3 integrability compared to AdS_5 and AdS_4 :

- massless modes coexist with massive We will consider only massive sector
- There are more then one dressing phase, and crossing and unitarity conditions seem to be not enough to fix dressing phases for S-matrix

Mixed flux classical integrability

ACTION:

The only WZ term which can be added to MT action in the case of $\mathbb{Z}_4$ permutation supercoset $G \times G/H$ is $(\kappa^2 + \chi^2 = 1)$ [A.Cagnazzo, K.Zarembo, 2012]

$$
\begin{aligned}
S_{mixed} &= \frac{1}{2}\int_{\mathcal{M}} (J_2 \wedge * J_2 + \kappa J_1 \wedge J_3) + \\
&+ \chi \int_{\mathcal{B}} Str(\frac{2}{3} J_2 \wedge J_2 \wedge J_2 + J_1 \wedge J_3 \wedge J_2 + J_3 \wedge J_1 \wedge J_2)
\end{aligned}
$$

The action is both κ-symmetric and integrable iff $\kappa^2 + \chi^2 = 1$. But $\mathbb{Z}_4$ symmetry is broken to $\mathbb{Z}_2$. Change notations:

$$
g_L \oplus g_R \ni x = \left(\begin{array}{c||c} x_L & 0 \\ \hline\hline 0 & x_R \end{array}\right), \quad \Omega_4(x) = \left(\begin{array}{c||c} x_R & 0 \\ \hline\hline 0 & (-)^F x_L \end{array}\right),
$$

$$
J_{L,R} = g_{L,R}^{-1} d g_{L,R}
$$

Mixed flux classical integrability

In these notations we introduce non dynamical variable - matrix W:

$$W = \left(\begin{array}{c||c} +1 & 0 \\ \hline\hline 0 & -1 \end{array}\right), \quad \Omega_4(W) = -W$$

and define the supertrace for $x \in \mathfrak{g}_L \oplus \mathfrak{g}_R$ as

$$Str(x) = Str(x_L) + Str(x_R)$$

Action

$$\begin{aligned} S_{mixed} =& \frac{1}{2}\int_{\mathcal{M}} \mathrm{Str}\left(J_2 \wedge *J_2 + \kappa J_1 \wedge J_3\right) \\ & + \chi \int_{\mathcal{B}} \mathrm{Str}\, W \left(\frac{2}{3} J_2 \wedge J_2 \wedge J_2 + J_1 \wedge J_3 \wedge J_2 + J_3 \wedge J_1 \wedge J_2\right). \end{aligned}$$

is now $\mathbb{Z}_4$ invariant, but it is not a physical symmetry.
($\Omega_4(W) = -W$ is the same as $\chi \to -\chi$).

Mixed flux classical integrability

LAX CONNECTION $A(x)$:

We look for solutions A of $dA + A \wedge A = 0$ with the ansatz

$$A = J_0 + \gamma_2 J_2 + \gamma_* * J_2 + \gamma_1 J_1 + \gamma_3 J_3$$

where $\gamma_i = \alpha_i I + \beta_i W$ are matrices. In terms of

$$\delta_*(\chi) = \alpha_* + \beta_* = \chi \pm \sqrt{\delta_2^2(\chi) - \kappa^2},$$
$$\delta_1(\chi) = \alpha_1 + \beta_1 = \pm\sqrt{\delta_2(\chi) - \kappa\delta_*(\chi)},$$
$$\delta_3(\chi) = \alpha_3 + \beta_3 = \pm\sqrt{\delta_2(\chi) + \kappa\delta_*(\chi)},$$

the most general Lax connection

$$A(x) = \frac{1}{2}\left((\delta_i(\chi) + \delta_i(-\chi))J_i + (\delta_i(\chi) - \delta_i(-\chi))WJ_i\right)$$

$i = 0, 1, 2, 3, *$ and $\delta_0(\chi) = 1$.

Mixed flux classical integrability

LAX CONNECTION $A(x)$:

Flat connection is not unique: $A \to A' = hAh^{-1} - dhh^{-1}$.
Compared to [CZ] we use different parametrization for spectral parameter of $A(x)$ applying gauge transformation. We require:

- It has standard form at $\chi \to 0$.
- Gauge out J_0: $A \to a = g(A - J)g^{-1}$ and require $a \sim \frac{\text{Noeter current}}{x}$ for $x \to \infty$
- $A(1/x) = \Omega_4(A(x))$
- The poles should coincide with the ones obtained from BE analysis

It gives:

$$\delta_2 = \frac{\left(x^2+1\right)\kappa}{\left(x^2-1\right)\kappa - 2x\chi}, \qquad \delta_1 = (x+1)\sqrt{\frac{\kappa}{\left(x^2-1\right)\kappa - 2x\chi}},$$

$$\delta_* = -\frac{2x\kappa}{(x\kappa-\chi)^2 - 1}, \qquad \delta_3 = (x-1)\sqrt{\frac{\kappa}{\left(x^2-1\right)\kappa - 2x\chi}}.$$

Mixed flux classical integrability

LAX CONNECTION $A(x)$:

The poles of $A(x)$ are shifted compared to pure RR case and located at $x = s = \hat{s}_+, -1/s = \hat{s}_-, -s = \check{s}_+, 1/s = \check{s}_-$ where

$$s = s(\chi) = \sqrt{\frac{1+\chi}{1-\chi}},$$

with residues

$$A\left(x \simeq s\right) = \frac{1}{2}\left(\mathbb{I} + W\right)\frac{s}{x-s}\left(J_2 + *J_2\right) + \cdots$$
$$A\left(x \simeq -s^{-1}\right) = \frac{1}{2}\left(\mathbb{I} + W\right)\frac{-s^{-1}}{x+s^{-1}}\left(J_2 - *J_2\right) + \cdots$$
$$A\left(x \simeq -s\right) = \frac{1}{2}\left(\mathbb{I} - W\right)\frac{-s}{x+s}\left(J_2 - *J_2\right) + \cdots$$
$$A\left(x \simeq s^{-1}\right) = \frac{1}{2}\left(\mathbb{I} - W\right)\frac{s^{-1}}{x-s^{-1}}\left(J_2 + *J_2\right) + \cdots$$

Mixed flux classical integrability

LAX CONNECTION $A(x)$:

The chosen parametrization of Lax connection degenerates in the pure NS-NS limit $(\chi \to 1, \kappa \to 0)$: only non zero coefficient is $\delta_* = 1$.
BUT, if we want to avoid this, we can do it preserving a general parametrization of δ_2 yields

$$W\delta_* = 1 - \delta_2,\ \ \delta_1 = \delta_3 = \pm\sqrt{\delta_2},$$
$$\text{or} \quad W\delta_* = 1 + \delta_2, \delta_1 = -\delta_3 = \pm\sqrt{\delta_2}.$$

without (in general) degeneracy of Lax connection. For example:

$$\delta_2 = \frac{1 + x^2 - \chi^2}{(x-\chi)^2 - 1}, \qquad \delta_1 = \frac{x+\kappa}{\sqrt{(x-\chi)^2 - 1}},$$
$$\delta_* = -\frac{2x}{(x-\chi)^2 - 1}, \qquad \delta_3 = \frac{x-\kappa}{\sqrt{(x-\chi)^2 - 1}}.$$

Finite gap equations

Because of restored $\mathbb{Z}_4$ symmetry a standard procedure for derivation of finite gap equations can be applied.

[A.B.,B.Stefanski,K.Zarembo, 2009], [K.Zarembo, 2010]

With quasimomenta $p_i(x,\chi)$ monodromy matrix
$M(x,\chi) = P\exp\int_0^{2\pi} A_\sigma(x,\chi) = U^{-1}(x,\chi)\exp(p_i(x,\chi)H_i)U(x,\chi)$,
- gauge invariant and defined up to Weyl group transformations.
By construction, M is given by two blocks, therefore also p : $\hat{p}$ with poles at $\hat{s}_\pm$,and $\check{p}$ with poles at $\check{s}_\pm$

Encircling a branch point $p_l(x) \to p_l(x) - A_{lm}p_m(x) + 2\pi n_{l,i}$ with $A_{ll} = 0$ for fermionic roots - log branch points, $A_{ll} = 2$ for bosonic roots - $\sqrt{}$ branch points.

$$A_{lm}\not{p}_m = 2\pi n_{l,i}, \quad x \in C_{l,i}$$

Finite gap equations

Close to the poles

$$\hat{p}_l(x \simeq \hat{s}_\pm) \to \pm \frac{\hat{s}_\pm}{2} \frac{\hat{\kappa}_l \pm 2\pi \hat{m}_l}{x - \hat{s}_\pm} + \cdots ,$$

$$\check{p}_l(x \simeq \check{s}_\pm) \to \mp \frac{\hat{s}_\pm}{2} \frac{\check{\kappa}_l \mp 2\pi \check{m}_l}{x + \hat{s}_\pm} + \cdots ,$$

where

$$\hat{s}_+ = s, \quad \hat{s}_- = -1/s, \quad \check{s}_+ = -s, \quad \check{s}_- = 1/s.$$

There should exist a matrix $S_{ml} : \Omega_4(H_m) = H_m S_{ml}$ and $p_l(1/x) = S_{lm} p_m(x)$

Virasoro constraints are not modified by WZ term

$$(\kappa_l \pm 2\pi m_l) A_{lk} (\kappa_k \pm 2\pi m_k) = 0.$$

Finite gap equations

Using the residues analysis one can show that spectral rep. of quasimomenta in terms of discontinuities at the cuts

$$\hat{p}_l(x) = \frac{\frac{x}{\kappa}\left(2\pi\chi\hat{m}_l + \hat{\kappa}_l\right) + 2\pi\hat{m}_l}{(x-s)\left(x+s^{-1}\right)} + \int_{C_l} dy \frac{\hat{\rho}_l(y)}{x-y} + \int_{1/C_l} dy \frac{\hat{\tilde{\rho}}_l(y)}{x-y},$$

$$\check{p}_l(x) = \frac{\frac{x}{\kappa}\left(2\pi\chi\check{m}_l - \check{\kappa}_l\right) - 2\pi\check{m}_l}{(x+s)\left(x-s^{-1}\right)} + \int_{C_l} dy \frac{\check{\rho}_l(y)}{x-y} + \int_{1/C_l} dy \frac{\check{\tilde{\rho}}_l(y)}{x-y},$$

similar for $\hat{p}_l(1/x), \check{p}_l(1/x)$...

Finite gap equations

$$\hat{p}_l(1/x) = \frac{\frac{x}{\kappa}\left(2\pi\chi\hat{m}_l - \hat{\kappa}_l\right) - 2\pi\hat{m}_l}{\left(x - s^{-1}\right)\left(x + s\right)} - 2\pi\hat{m}_l +$$

$$\int_{1/C_l} dy \frac{\hat{\rho}_l(1/y)}{y} + \int_{C_l} dy \frac{\hat{\tilde{\rho}}_l(1/y)}{y} + \int_{1/C_l} dy \frac{\hat{\rho}_l(1/y)}{x-y} + \int_{C_l} dy \frac{\hat{\tilde{\rho}}_l(1/y)}{x-y},$$

Specifying the grading for $psl(1,1|2)$: $S = \sigma_1 \otimes \mathbb{S}_{lk}$, $A = \sigma_3 \otimes \mathbb{A}$ with

$$\mathbb{A} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathbb{S} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

we find

$$\hat{\kappa}_l = \mathbb{S}_{lm}\check{\kappa}_m, \quad \hat{m}_l = \mathbb{S}_{lm}\check{m}_m,$$

$$2\pi\hat{m}_l = -\int_{C_l} dy \frac{\hat{\rho}_l(y)}{y} + \mathbb{S}_{lm}\int_{C_m} dy \frac{\check{\rho}_m(y)}{y},$$

$$2\pi\check{m}_l = +\int_{C_l} dy \frac{\check{\rho}_l(y)}{y} - \mathbb{S}_{lm}\int_{C_m} dy \frac{\hat{\rho}_m(y)}{y},$$

Finite gap equations

$$
\begin{aligned}
2\pi\hat{n}_{k,i} =& \mathbb{A}_{kl}\frac{\frac{x}{\kappa}\left(2\pi\chi\hat{m}_l+\hat{\kappa}_l\right)+2\pi\hat{m}_l}{\left(x-s\right)\left(x+s^{-1}\right)} \\
&+\mathbb{A}_{kl}\fint_{C_l}dy\,\frac{\hat{\rho}_l(y)}{x-y}-\mathbb{A}_{kl}\mathbb{S}_{lm}\int_{C_m}\frac{dy}{y^2}\frac{\check{\rho}_m(y)}{x-1/y}, \\
2\pi\check{n}_{k,i} =& -\mathbb{A}_{kl}\mathbb{S}_{lm}\frac{\frac{x}{\kappa}\left(2\pi\chi\hat{m}_m-\hat{\kappa}_m\right)-2\pi\hat{m}_m}{\left(x-s^{-1}\right)\left(x+s\right)} \\
&-\mathbb{A}_{kl}\fint_{C_l}dy\,\frac{\check{\rho}_l(y)}{x-y}+\mathbb{A}_{kl}\mathbb{S}_{lm}\int_{C_m}\frac{dy}{y^2}\frac{\hat{\rho}_m(y)}{x-1/y},
\end{aligned}
$$

Using $\mathcal{P}_i=\frac{1}{4\pi}\int dy\,\frac{\rho_i(y)}{y}$ we finally get

$$
\begin{aligned}
&\hat{\kappa}=2\pi\mathcal{E}(1,0,1), \\
&\hat{m}=-2(\hat{\mathcal{P}}-S\check{\mathcal{P}})=2\left(-\hat{\mathcal{P}}_1+\check{\mathcal{P}}_1-\check{\mathcal{P}}_2,-\hat{\mathcal{P}}_2-\check{\mathcal{P}}_2,-\hat{\mathcal{P}}_3+\check{\mathcal{P}}_3-\check{\mathcal{P}}_2\right) \\
&\check{m}=+2(\check{\mathcal{P}}-S\hat{\mathcal{P}})=2\left(-\hat{\mathcal{P}}_1+\check{\mathcal{P}}_1+\hat{\mathcal{P}}_2,+\hat{\mathcal{P}}_2+\check{\mathcal{P}}_2,-\hat{\mathcal{P}}_3+\check{\mathcal{P}}_3+\hat{\mathcal{P}}_2\right),
\end{aligned}
$$

Finite gap equations

$$2\pi\hat{n}_{1,i} = -\int dy \frac{\hat{\rho}_2(y)}{x-y} - \int \frac{dy}{y^2}\frac{\check{\rho}_2(y)}{x-\frac{1}{y^2}},$$

$$2\pi\hat{n}_{2,i} = 2\fint dy \frac{\hat{\rho}_2(y)}{x-y} - \int dy \frac{\hat{\rho}_1(y)}{x-y} - \int dy \frac{\hat{\rho}_3(y)}{x-y} + \int \frac{dy}{y^2}\frac{\check{\rho}_1(y)}{x-\frac{1}{y}} + \int \frac{dy}{y^2}\frac{\check{\rho}_3(y)}{x-\frac{1}{y}}$$
$$- \frac{4\pi x}{(x-s)(x+s^{-1})}\left(\frac{1}{\kappa}\mathcal{E} - \frac{\chi}{\kappa}\mathcal{M}\right) + \frac{4\pi}{(x-s)(x+s^{-1})}\mathcal{M},$$

$$2\pi\hat{n}_{3,i} = -\int dy \frac{\hat{\rho}_2(y)}{x-y} - \int \frac{dy}{y^2}\frac{\check{\rho}_2(y)}{x-\frac{1}{y^2}},$$

$$2\pi\check{n}_{1,i} = +\int dy \frac{\check{\rho}_2(y)}{x-y} + \int \frac{dy}{y^2}\frac{\hat{\rho}_2(y)}{x-\frac{1}{y^2}},$$

$$2\pi\check{n}_{2,i} = -2\fint dy \frac{\check{\rho}_2(y)}{x-y} + \int dy \frac{\check{\rho}_1(y)}{x-y} + \int dy \frac{\check{\rho}_3(y)}{x-y} - \int \frac{dy}{y^2}\frac{\hat{\rho}_1(y)}{x-\frac{1}{y}} - \int \frac{dy}{y^2}\frac{\hat{\rho}_3(y)}{x-\frac{1}{y}}$$
$$- \frac{4\pi x}{(x+s)(x-s^{-1})}\left(\frac{1}{\kappa}\mathcal{E} + \frac{\chi}{\kappa}\mathcal{M}\right) + \frac{4\pi}{(x+s)(x-s^{-1})}\mathcal{M},$$

$$2\pi\check{n}_{3,i} = +\int dy \frac{\check{\rho}_2(y)}{x-y} + \int \frac{dy}{y^2}\frac{\hat{\rho}_2(y)}{x-\frac{1}{y^2}},$$

where

$$\mathcal{M} = \hat{\mathcal{P}}_1 - \check{\mathcal{P}}_1 + 2\check{\mathcal{P}}_2 + \hat{\mathcal{P}}_3 - \check{\mathcal{P}}_3.$$

Spin chain symmetry

Ground state [R.Borsato,O.Ohlsson Sax,A.Sfondrini,B.Stefanski,A.Torrielli, 2012,2013]

Sites are occupied by $(-\frac{1}{2}, \frac{1}{2})$ reps of $psu(1,1|2)$ with h.w. V.

Vacuum $|0\rangle = (V)^L \otimes (V)^L$ is preserved by 8 super- and 2 central charges $\{Q_i, \bar{Q}_i, S_i, \bar{S}_i, H, \bar{H}\}$, $i = 1, 2$,

$$\{Q_i, S_j\} = \delta_{ij} H, \quad \{\bar{Q}_i, \bar{S}_j\} = \delta_{ij} \bar{H}$$

They form $$psu(1|1)^2 \ltimes u(1)^2 \rtimes \overline{psu(1|1)}^2$$

Hamiltonian: $H = H + \bar{H}$,
Angular momentum of AdS_3: $M = H - \bar{H}$

Additionally there are outer automorphisms B_i acting on each $psu(1|1)^2$

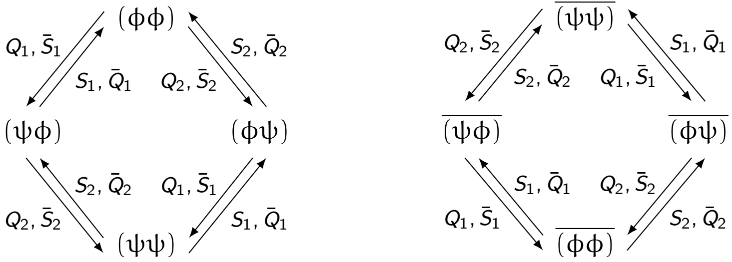
The full symmetry preserving the ground state

$$\left[u(1) \ltimes psu(1|1)^2\right]^2 \ltimes u(1)^2$$

Spin chain symmetry

Excitations

Excitations - states of $psu(1,1|2)^2$ - replace one or more ground state sites. Looking at the Hamiltonian one can select the lightest set. They form the bifundamental reps of $psu(1|1)^2 \oplus \overline{psu(1|1)}^2$.



Left/right algebra action on right/left states leads to spin length change and to two additional central extensions $\{Q_i, \bar{Q}_j\} = \delta_{ij}\mathcal{P}$, $\{S_i, \bar{S}_j\} = \delta_{ij}\mathcal{P}^\dagger$, giving the final symmetry

$\left[u(1) \ltimes psu(1|1)^2\right]^2 \ltimes u(1)^4$

Pure RR S-matrix

The all loop massive modes scattering S-matrix of $AdS_3 \times S^3 \times T^4$ with pure RR flux is $\mathcal{S} = \mathbb{S}_{su(1|1)^2} \check{\otimes} \mathbb{S}_{su(1|1)^2}$.

[R.Borsato,O.Ohlsson Sax,A.Sfondrini,B.Stefanski,A.Torrielli, 2012,2013]

follows from the symmetry $[\mathcal{S}, \{Q\}] = 0$ and YB equation $\mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23} = \mathcal{S}_{23}\mathcal{S}_{13}\mathcal{S}_{12}$:

$$\mathbb{S}|\phi\phi'\rangle = L_1|\phi\phi'\rangle, \ \mathbb{S}|\phi\psi'\rangle = L_3|\phi\psi'\rangle + L_5|\psi\phi'\rangle,$$

$$\mathbb{S}|\psi\psi'\rangle = \Lambda_1|\psi\psi'\rangle, \ \mathbb{S}|\psi\phi'\rangle = \Lambda_3|\psi\phi'\rangle + \Lambda_5|\phi\psi'\rangle,$$

$$\mathbb{S}|\phi\bar{\psi}'\rangle = L_6|\phi\bar{\psi}'\rangle, \ \mathbb{S}|\phi\bar{\phi}'\rangle = L_2|\phi\bar{\phi}'\rangle + L_4|\psi\bar{\psi}'\rangle,$$

$$\mathbb{S}|\psi\bar{\phi}'\rangle = \Lambda_6|\psi\bar{\phi}'\rangle, \ \mathbb{S}|\psi\bar{\psi}'\rangle = \Lambda_2|\psi\bar{\psi}'\rangle + \Lambda_4|\phi\bar{\phi}'\rangle,$$

The functions $L_1, L_3, L_5, \Lambda_1, \Lambda_3, \Lambda_5$ are fixed by the symmetry up to a common scalar functions σ, and $L_2, L_4, L_6, \Lambda_2, \Lambda_4, \Lambda_6$ – up to a common scalar functions $\bar{\sigma}$.

Mixed flux: Hoare -Tseytlin S-matrix

[B.Hoare, A.Tseytlin 2013]

WZ term doesn't change the S-matrix symmetry

$$\left(u(1) \ltimes psu(1|1)^2\right)^2 \ltimes u(1)^4$$

but change its representation
4 of the 6 central charges are physical and responsible for dispersion relation. Tree level scattering : two different reps of excitations

$$E = \sqrt{M(p,\chi)^2 + 16h^2\kappa^2 \sin^2 \frac{p}{2}}$$

$$\hat{M}(p,\chi)^2 = (1 + 4\chi h f(p))^2, \qquad \check{M}(p,\chi)^2 = (1 - 4\chi h f(p))^2$$

Conjectured $f(p) = \sin(p/2)$ was later corrected to $f(p) = p/2$ by calculation of giant magnon scattering.

Hoare -Tseytlin S-matrix

Dispersion relation

Two different masses for $\chi \neq 0$ lead to the split of Zhukowski variables into two branches $\hat{x}^{\pm}, \check{x}^{\pm}$

$$\frac{\hat{x}^{+}}{\hat{x}^{-}} = e^{+ip}, \qquad \frac{\check{x}^{+}}{\check{x}^{-}} = e^{+ip}$$

shortening condition

$$\hat{x}^{+} + \frac{1}{\hat{x}^{+}} - \hat{x}^{-} - \frac{1}{\hat{x}^{-}} = \frac{i\hat{M}}{\kappa h} = \frac{i}{\kappa h}\left(1 + 2\frac{\chi}{\kappa}\hat{P}\right),$$

$$\check{x}^{+} + \frac{1}{\check{x}^{+}} - \check{x}^{-} - \frac{1}{\check{x}^{-}} = \frac{i\check{M}}{\kappa h} = \frac{i}{\kappa h}\left(1 - 2\frac{\chi}{\kappa}\check{P}\right),$$

$$\hat{P} = \kappa h p = \check{P}.$$

Dispersion relation

$$\hat{E} = -i\kappa h\left(\left(\hat{x}^{+} - \hat{x}^{-}\right) - \left(\frac{1}{\hat{x}^{+}} - \frac{1}{\hat{x}^{-}}\right)\right),$$

$$\check{E} = -i\kappa h\left(\left(\check{x}^{+} - \check{x}^{-}\right) - \left(\frac{1}{\check{x}^{+}} - \frac{1}{\check{x}^{-}}\right)\right).$$

HT S-matrix

Crossing relation

Dressing phases $\sigma, \bar{\sigma}$ satisfy crossing equations

$$\sigma^2(x^\pm, y^\pm)\,\bar{\sigma}^2(x^\pm, 1/y^\pm) = \left(\frac{x^+}{x^-}\right)^2 \frac{(x^- - y^+)^2}{(x^- - y^-)(x^+ - y^+)} \frac{1 - \frac{1}{x^- y^+}}{1 - \frac{1}{x^+ y^-}},$$

$$\sigma^2(x^\pm, 1/\bar{y}^\pm)\,\bar{\sigma}^2(x^\pm, \bar{y}^\pm) = \left(\frac{x^+}{x^-}\right)^2 \frac{\left(1 - \frac{1}{x^- \bar{y}^-}\right)\left(1 - \frac{1}{x^+ \bar{y}^+}\right)}{\left(1 - \frac{1}{x^+ \bar{y}^-}\right)^2} \frac{x^- - \bar{y}^+}{x^+ - \bar{y}^-}.$$

here $x^\pm, y^\pm$ are variables of the same kind (either "hat" or "check"), and $\bar{y}$ - of opposite kind to y.

Bethe equations

Nested CBA was constructed for pure RR flux.

[R.Borsato,O.Ohlsson Sax,A.Sfondrini,B.Stefanski,A.Torrielli, 2012,2013]

Since the S-matrix for mixed flux has the same symmetry the construction generalizes to mixed flux.

Momentum carrying Bethe roots $\hat{x}_{2,k}, \check{x}_{2,k}$ and four sets of auxiliary roots $\hat{x}_{1,k}, \check{x}_{1,k}, \hat{x}_{3,k}, \check{x}_{3,k},$.

Remark: Spin chain interpretations seems not so natural because of non periodic dispersion relations, but formally possible. ABA should be done.

Bethe equations

$$y \qquad x \quad = \prod_j \frac{y_k - x_j^+}{y_k - x_j^-}$$

$$y \qquad x \quad = \prod_j \frac{1 - 1/y_k x_j^-}{1 - 1/y_k x_j^+}$$

$$\textcircled{\hat{2}} = \left(\frac{\hat{x}_{2,k}^+}{\hat{x}_{2,k}^-} \right)^L, \quad \textcircled{\check{2}} = \left(\frac{\check{x}_{2,k}^-}{\check{x}_{2,k}^+} \right)^L,$$

$$\otimes \quad = 1,$$

$$x_k \qquad x_j \quad = \prod_j \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1 - 1/x_k^{+s} x_j^{-s}}{1 - 1/x_k^{-s} x_j^{+s}} \sigma^{2s}(x_k, x_j)$$

$$x_k \qquad x_j \quad = \prod_j \frac{1 - 1/x_k^+ \bar{x}_j^+}{1 - 1/x_k^- \bar{x}_j^-} \frac{1 - 1/x_k^{+s} \bar{x}_j^{-s}}{1 - 1/x_k^{-s} \bar{x}_j^{+s}} \bar{\sigma}^{2s}(x_k, \bar{x}_j)$$

$$s = + \text{ for } \hat{2}, \; s = - \text{ for } \check{2}$$

Bethe equations

$$y \;\; x \quad = \prod_j \frac{y_k - x_j^+}{y_k - x_j^-}$$

$$y \;\; x \quad = \prod_j \frac{1 - 1/y_k x_j^-}{1 - 1/y_k x_j^+}$$

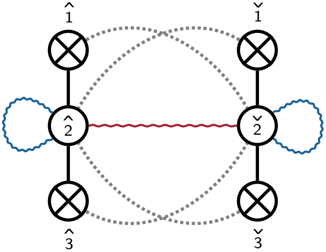
$$\hat{2} = \left(\frac{\hat{x}_{2,k}^+}{\hat{x}_{2,k}^-}\right)^L, \quad \check{2} = \left(\frac{\check{x}_{2,k}^-}{\check{x}_{2,k}^+}\right)^L,$$

$$\otimes = 1,$$

$$x_k \;\; x_j \quad = \prod_j \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1 - 1/x_k^{+s} x_j^{-s}}{1 - 1/x_k^{-s} x_j^{+s}} \sigma^{2s}(x_k, x_j)$$

$$x_k \;\; x_j \quad = \prod_j \frac{1 - 1/x_k^+ \bar{x}_j^+}{1 - 1/x_k^- \bar{x}_j^-} \frac{1 - 1/x_k^{+s} \bar{x}_j^{-s}}{1 - 1/x_k^{-s} \bar{x}_j^{+s}} \bar{\sigma}^{2s}(x_k, \bar{x}_j)$$

$$s = + \text{ for } \hat{2}, \; s = - \text{ for } \check{2}$$



NSNS limit of Bethe equations

Coming back from Zhukowski variables to momenta in the limit $\kappa \to 0, \chi \to 1$

$$
\begin{aligned}
\hat{x}_2^{\pm} &= (ih\kappa f_+^{\mp}(\hat{p}))^{-1}, \quad \check{x}_2^{\pm} = (ih\kappa f_-^{\mp}(\check{p}))^{-1} \\
\hat{x}_2^{\pm} &= ih\kappa f_+^{\pm}(\hat{q}), \quad \check{x}_2^{\pm} = ih\kappa f_-^{\pm}(\check{q}) \qquad f_{\pm}^{\pm}(x) = \frac{1 - e^{\pm ix}}{1 \pm 2hx}
\end{aligned}
$$

Equations for $\check{x}_2, \hat{x}_2$ split into 2 each. Auxiliary equations become identities. One of the four equations:

$$
e^{i\widehat{p}_k(L-\tilde{K})} = \prod_{j=1,\neq k}^{\widehat{K}_2'} \frac{e^{i\widehat{p}_k} f_+^+(\widehat{p}_j) - f_+^+(\widehat{p}_k)}{e^{i\widehat{p}_j} f_+^+(\widehat{p}_k) - f_+^+(\widehat{p}_j)} \prod_{j=1}^{\widehat{K}_2''} \frac{1 - f_+^-(\widehat{p}_k)/f_+^-(\widehat{q}_j)}{1 - f_+^+(\widehat{p}_k)/f_+^+(\widehat{q}_j)} \times
$$

$$
\prod_{j=1}^{\check{K}_2''} \frac{1 - f_+^-(\widehat{p}_k)/f_-^+(\check{q}_j)}{1 - f_+^+(\widehat{p}_k)/f_-^-(\check{q}_j)} \frac{1 + f_+^-(\widehat{p}_k)/f_-^-(\check{q}_j)}{1 + f_+^+(\widehat{p}_k)/f_-^+(\check{q}_j)} \times
$$

$$
\prod_{j=1,\neq k}^{\widehat{K}_2'} \sigma^2(\widehat{p}_k, \widehat{p}_j) \prod_{j=1}^{\widehat{K}_2''} \sigma^2(\widehat{p}_k, \widehat{q}_j) \prod_{j=1}^{\check{K}_2'} \widetilde{\sigma}^2(\widehat{p}_k, \check{p}_j) \prod_{j=1}^{\check{K}_2''} \widetilde{\sigma}^2(\widehat{p}_k, \check{q}_j)
$$

Dressing phase at tree level

We assume that dressing phase in the leading (tree level) order is given by AFS phase

$$-\frac{i}{\kappa h}\log\sigma_{\mathrm{AFS}}(x^{\pm},y^{\pm}) = \chi(x^+,y^+)-\chi(x^+,y^-)-\chi(x^-,y^+)+\chi(x^-,y^-)$$

$$\chi(x,y) = \left(y+\frac{1}{y}-x-\frac{1}{x}\right)\log\left(1-\frac{1}{xy}\right)$$

The phase $\bar{\sigma}$ can then be obtained from the crossing relation:

$$-\frac{i}{\kappa h}\log\bar{\sigma}(\hat{x}^{\pm},\check{y}^{\pm}) = \bar{\chi}(\hat{x}^+,\check{y}^+)-\bar{\chi}(\hat{x}^+,\check{y}^-)-\bar{\chi}(\hat{x}^-,\check{y}^+)+\bar{\chi}(\hat{x}^-,\check{y}^-)$$

$$\bar{\chi}(\hat{x},\check{y}) = \left(\check{y}+\frac{1}{\check{y}}-\hat{x}-\frac{1}{\hat{x}}\right)\log\left(1-\frac{1}{\hat{x}\check{y}}\right)-\frac{\chi}{\kappa}\left(2\,\mathrm{Li}_2\left(\frac{1}{\hat{x}\check{y}}\right)-\log\hat{x}\,\log\check{y}\right)$$

Finite gap from Bethe equations

Consider Bethe equations in the scaling limit $L \approx K_i \gg 1$,and take $h \gg 1$. Bethe roots condense to cuts : $\hat{x}_{i,k} \to \hat{C}_i, \check{x}_{i,k} \to \check{C}_i$
Expanding log of BE at large x we obtain global charges of solutions: J, K - a.m. on S^3, S - a.m. on AdS_3, D - global energy:

$$
\begin{aligned}
D &= +\check{K}_2 + \frac{1}{2}\left(\hat{K}_1 + \hat{K}_3 - \check{K}_1 - \check{K}_3\right) + L + \delta D,\\
J &= -\hat{K}_2 + \frac{1}{2}\left(\hat{K}_1 + \hat{K}_3 - \check{K}_1 - \check{K}_3\right) + L,\\
K &= -\hat{K}_2 + \frac{1}{2}\left(\hat{K}_1 + \hat{K}_3 + \check{K}_1 + \check{K}_3\right) - 2\frac{\chi}{\kappa}(\hat{P} + \check{P}),\\
S &= -\check{K}_2 + \frac{1}{2}\left(\hat{K}_1 + \hat{K}_3 + \check{K}_1 + \check{K}_3\right).
\end{aligned}
$$

It can be shown that the anomalous dimension

$$
\delta D = 2\kappa h \mathcal{Q}_2 + 2\frac{\chi}{\kappa}(\hat{P} - \check{P})
$$

Finite gap from Bethe equations

Shortening condition at strong coupling can be solved

$$\hat{x}^{\pm} = x \pm \frac{i}{2}\hat{\alpha}(x) + \mathcal{O}(1/h^2), \qquad \check{x}^{\pm} = x \pm \frac{i}{2}\check{\alpha}(x) + \mathcal{O}(1/h^2)$$

$$\hat{\alpha}(x) = \frac{1}{\kappa h}\frac{x^2}{(x-s)(x+s^{-1})}, \qquad \check{\alpha}(x) = \frac{1}{\kappa h}\frac{x^2}{(x+s)(x-s^{-1})}$$

With densities defined as

$$\hat{\rho}_i(x) = \sum_k \alpha(\hat{x}_{i,k})\delta(x-\hat{x}_{i,k}), \qquad \check{\rho}_i(x) = \sum_k \check{\alpha}(\check{x}_{i,k})\delta(x-\check{x}_{i,k})$$

log of Bethe equations reproduce the finite gap integral equations.

Finite gap from Bethe equations

Coefficients $\mathcal{E}$ and $\mathcal{M}$ are expressed through

$$\hat{\mathcal{P}}_m = \frac{1}{4\pi}\int \frac{dy}{y}\hat{\rho}_m(y), \qquad \hat{\mathcal{E}}_m = \frac{\kappa}{4\pi}\int \frac{dy}{y^2}\hat{\rho}_m(y),$$

$$\check{\mathcal{P}}_m = \frac{1}{4\pi}\int \frac{dy}{y}\check{\rho}_m(y), \qquad \check{\mathcal{E}}_m = \frac{\kappa}{4\pi}\int \frac{dy}{y^2}\check{\rho}_m(y).$$

$$\mathcal{M} = +\hat{\mathcal{P}}_1 + \hat{\mathcal{P}}_3 - \check{\mathcal{P}}_1 + 2\check{\mathcal{P}}_2 - \check{\mathcal{P}}_3$$

$$\mathcal{E} = \mathcal{L} - \hat{\mathcal{E}}_1 + 2\hat{\mathcal{E}}_2 - \hat{\mathcal{E}}_3 + \check{\mathcal{E}}_1 + \check{\mathcal{E}}_3 - \chi\left(\hat{\mathcal{P}}_1 - 2\hat{\mathcal{P}}_2 + \hat{\mathcal{P}}_3 + \check{\mathcal{P}}_1 + \check{\mathcal{P}}_3\right)$$

with $\mathcal{L} = L/\sqrt{\lambda}$.

Anomalous dimension

$$\frac{\delta D}{\sqrt{\lambda}} = 2(\hat{\mathcal{E}}_2 + \check{\mathcal{E}}_2) + 2\chi(\hat{\mathcal{P}}_2 - \check{\mathcal{P}}_2)$$

total world sheet momentum

$$p_{\text{total}} = 4\pi(\hat{\mathcal{P}}_2 + \check{\mathcal{P}}_2)$$

Finite gap in fundamental rep

Introducing the resolvents ($\tilde{}$ stands for either $\hat{}$ or $\check{}$)

$$G_{\tilde{a}}(x) = \sum_{k=1}^{K_{\tilde{a}}} \frac{\tilde{\alpha}(x_{\tilde{a},k})}{x - x_{\tilde{a},k}}, \quad H_{\tilde{a}}(x) = \sum_{k=1}^{K_{\tilde{a}}} \frac{\tilde{\alpha}(x)}{x - x_{\tilde{a},k}},$$

$$\bar{G}_{\tilde{a}}(x) = G_{\tilde{a}}(1/x), \quad \bar{H}_{\tilde{a}}(x) = H_{\tilde{a}}(1/x)$$

one can define 8 quasimomenta

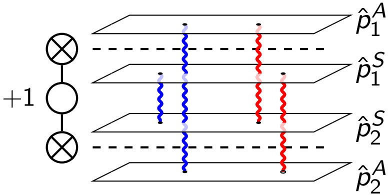
$$\begin{aligned} \hat{p}_1^A - \hat{p}_1^S &= 2\pi n_{\hat{1}}, & \check{p}_2^S - \check{p}_2^A &= 2\pi n_{\check{3}}, \\ \hat{p}_1^S - \hat{p}_2^S &= 2\pi n_{\hat{2}}, & \check{p}_2^A - \check{p}_1^A &= 2\pi n_{\check{2}}, \\ \hat{p}_2^S - \hat{p}_2^A &= 2\pi n_{\hat{3}}, & \check{p}_1^A - \check{p}_1^S &= 2\pi n_{\check{1}}. \end{aligned}$$

and find them in terms of resolvents, e.g.

$$\begin{aligned} \hat{p}_1^A(x) &= +\bar{H}_{\check{2}} - H_{\hat{1}} - \bar{H}_{\check{1}} \\ &+x \frac{\frac{\chi}{\kappa}(G_{\hat{2}}(0) - G_{\check{2}}(0)) + G'_{\hat{2}}(0) + G'_{\check{2}}(0)}{(x-s)(x+s^{-1})} - \frac{1}{2}\frac{\frac{4\pi}{\kappa}\mathcal{L}x}{(x-s)(x+s^{-1})}, \end{aligned}$$

Finite gap in fundamental rep

Synchronized poles correspond to excitations. There are 8 bosonic (blue) and fermionic (red) different polarizations of excitations.



8 sheets disconnected Riemann sirface are related by $x \to 1/x$.

$$\begin{aligned} \hat{p}_1^A(1/x) &= \check{p}_1^A(x), \qquad & \hat{p}_1^S(1/x) &= \check{p}_1^S(x) + G_{\hat{2}}(0) + G_{\check{2}}(0), \\ \hat{p}_2^A(1/x) &= \check{p}_2^A(x), \qquad & \hat{p}_2^S(1/x) &= \check{p}_2^S(x) - G_{\hat{2}}(0) - G_{\check{2}}(0). \end{aligned} \tag{1}$$

Semiclassical quantization

For semiclassics we add poles to quasimomenta in analytically consistent way $p_i(x) + \delta p_i(x)$. Poles will shift the macroscopic cuts:

$$(p_i + \delta p_i)^+ - (p_j + \delta p_j)^- = 2\pi n, \quad x \in \mathcal{C}_{ij},$$

This fixes position of microscopic cuts (poles) to the leading order

$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n, \quad |x_n^{ij}| > 1,$$

and along the macroscopic cuts the perturbation satisfies

$$(\delta p_i)^+ - (\delta p_j)^- = 0, \quad x \in \mathcal{C}_n^{ij}.$$

Possible excitations

$$\begin{aligned}
\mathrm{AdS}_3 : &\quad (\hat{p}_1^A, \hat{p}_2^A), (\check{p}_2^A, \check{p}_1^A) \\
\mathrm{S}^3 : &\quad (\hat{p}_1^S, \hat{p}_2^S), (\check{p}_2^S, \check{p}_1^S) \\
\mathrm{Fermionic} : &\quad (\hat{p}_1^A, \hat{p}_2^S), (\hat{p}_1^S, \hat{p}_2^A), (\check{p}_2^A, \check{p}_1^S), (\check{p}_2^S, \check{p}_1^A).
\end{aligned}$$

Semiclassical quantization

Let N_n^{ij} - the number of excitations with mode number n between the sheets p_i and p_j, and $N_{ij} = \sum_n N_n^{ij}$. The energy shift

$$\delta D = \delta\Delta + \sum_{\mathrm{AdS}_3} N_{ij} + \frac{1}{2} \sum_{\text{fermions}} N_{ij}$$

From finite gap equations one can find asymptotic $x \to \infty$ of quasimomenta shifts

$$\delta \left(\begin{array}{c} \hat{p}_1^A \\ \hat{p}_2^A \\ \hline \hat{p}_1^S \\ \hat{p}_2^S \\ \hline\hline \check{p}_1^A \\ \check{p}_2^A \\ \hline \check{p}_1^S \\ \check{p}_2^S \end{array} \right) = \frac{1}{\kappa h x} \left(\begin{array}{r} -\frac{1}{2}\delta\Delta - N_{\hat{1}\hat{2}}^{AA} - N_{\hat{1}\hat{2}}^{AS} \\ +\frac{1}{2}\delta\Delta + N_{\hat{1}\hat{2}}^{AA} + N_{\hat{1}\hat{2}}^{SA} \\ \hline + N_{\hat{1}\hat{2}}^{SS} + N_{\hat{1}\hat{2}}^{SA} \\ - N_{\hat{1}\hat{2}}^{SS} - N_{\hat{1}\hat{2}}^{AS} \\ \hline\hline +\frac{1}{2}\delta\Delta + N_{\check{2}\check{1}}^{AA} + N_{\check{2}\check{1}}^{AS} \\ -\frac{1}{2}\delta\Delta - N_{\check{2}\check{1}}^{AA} - N_{\check{2}\check{1}}^{SA} \\ \hline - N_{\check{2}\check{1}}^{SS} - N_{\check{2}\check{1}}^{SA} \\ + N_{\check{2}\check{1}}^{SS} + N_{\check{2}\check{1}}^{AS} \end{array} \right).$$

Semiclassical quantization

Residues of p_i are given by

$$\underset{x=x_n^{12}}{\text{res}}\,\hat{p}_i^A = -(\delta_{1i}-\delta_{2i})\hat{\alpha}(x_n^{12})N_n^{12},\ \underset{x=x_n^{12}}{\text{res}}\,\hat{p}_i^S \quad = +(\delta_{1i}-\delta_{2i})\hat{\alpha}(x_n^{12})N_n^{12},$$
$$\underset{x=x_n^{12}}{\text{res}}\,\check{p}_i^A = +(\delta_{1i}-\delta_{2i})\check{\alpha}(x_n^{12})N_n^{12},\ \underset{x=x_n^{12}}{\text{res}}\,\check{p}_i^S \quad = -(\delta_{1i}-\delta_{2i})\check{\alpha}(x_n^{12})N_n^{12}.$$

Poles of quasi-momenta δp are synchronized

$$\delta(\hat{p}_1^A,\hat{p}_2^A|\hat{p}_1^S,\hat{p}_2^S||\check{p}_1^A,\check{p}_2^A|\check{p}_1^S,\check{p}_2^S) \simeq \begin{cases} +s\frac{(\delta\alpha_+,\delta\beta_+|\delta\alpha_+,\delta\beta_+||0,0,0,0)}{x-s} \\ -\frac{1}{s}\frac{(\delta\alpha_-,\delta\beta_-|\delta\alpha_-,\delta\beta_-||0,0,0,0)}{x+1/s} \\ +s\frac{(0,0,0,0||\delta\alpha_-,\delta\beta_-|\delta\alpha_-,\delta\beta_-)}{x+s} \\ -\frac{1}{s}\frac{(0,0,0,0||\delta\alpha_+,\delta\beta_+|\delta\alpha_+,\delta\beta_+)}{x-1/s} \end{cases}$$

and satisfy $\mathbb{Z}_4$ symmetry

$$\delta\hat{p}_1^A(1/x) = \delta\check{p}_1^A(x), \qquad \delta\hat{p}_1^S(1/x) = \delta\check{p}_1^S(x),$$
$$\delta\hat{p}_2^A(1/x) = \delta\check{p}_2^A(x), \qquad \delta\hat{p}_2^S(1/x) = \delta\check{p}_2^S(x).$$

Semiclassical quantization

BMN string:

BMN solution is the simplest one - no cuts. The classical quasi-momenta is given by

$$p_l(x) = (p(x), -p(x)|p(x), -p(x)||p(1/x), -p(1/x), p(1/x), -p(1/x))$$

$$p(x) = \frac{2\pi x \mathcal{J}}{\kappa(x-s)(x-s^{-1})},$$

Position of the poles for, e.g. AdS excitations, are fixed by

$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n, \quad |x_n^{ij}| > 1, \quad x_n^{\hat{\imath}\hat{\jmath}} = \frac{\mathcal{J} + \chi n + \sqrt{\mathcal{J}^2 + 2\chi \mathcal{J} n + n^2}}{\kappa n}$$

Ansatz for perturbed quasimomenta

$$\delta \hat{p}_1^A(x) = +\frac{s\delta\alpha}{x-s} - \frac{(1/s)\delta\alpha}{x+1/s} - \sum_n N^n_{\hat{1}\hat{2}} \frac{\hat{\alpha}(x^n_{\hat{1}\hat{2}})}{x - x^n_{\hat{1}\hat{2}}} + \sum_n N^n_{\check{1}\check{2}} \frac{\check{\alpha}(x^n_{\check{1}\check{2}})}{1/x - x^n_{\check{1}\check{2}}}$$

$$+ \hat{a}_1^A, \quad \delta \hat{p}_1^S(x) = \frac{s\delta\alpha_+}{x-s} - \frac{(1/s)\delta\alpha_-}{x+1/s},$$

Semiclassical quantization

BMN string:

and

$$\delta\hat{p}_2^A(x) = -\delta\hat{p}_1^A(x), \qquad \delta\check{p}_i^A(x) = \delta\hat{p}_i^A(1/x),$$
$$\delta\hat{p}_2^S(x) = -\delta\hat{p}_1^S(x), \qquad \delta\check{p}_i^S(x) = \delta\hat{p}_i^S(1/x).$$

$$\hat{p}_1^A(x^n_{\hat{1}\hat{2}}) - \hat{p}_2^A(x^n_{\hat{1}\hat{2}}) = 2\pi n \qquad \check{p}_2^A(x^n_{\check{1}\check{2}}) - \check{p}_1^A(x^n_{\check{1}\check{2}}) = 2\pi n.$$

The constants a, α can be found from large x expansion, and residues at the poles, giving $\delta\Delta$ the energy fluctuation. In the same way other excitations can be added. Finally ($\xi = \frac{n}{\mathcal{J}}$)

$$\delta\Delta = \sum_{\text{all } ij} \sum_n \left(\hat{N}_{ij}^n \left(\sqrt{\xi^2 + 2\chi\xi + 1} - 1 \right) + \check{N}_{ij}^n \left(\sqrt{\xi^2 - 2\chi\xi + 1} - 1 \right) \right)$$

[D.E.Berenstein,J.M.Maldacena,H.S.Nastase 2002]

One loop correction to the dressing phases

To include one loop correction to the dressing phases

$$\sigma(\hat{x}^{\pm}, \hat{y}^{\pm}) = \exp\big(i\theta(\hat{x}^{\pm}, \hat{y}^{\pm})\big), \qquad \bar{\sigma}(\hat{x}^{\pm}, \check{y}^{\pm}) = \exp\big(i\bar{\theta}(\hat{x}^{\pm}, \check{y}^{\pm})\big)$$

$$\theta(\hat{x}^{\pm}, \hat{y}^{\pm}) = h\,\theta^{(0)}(\hat{x}^{\pm}, \hat{y}^{\pm}) + \theta^{(1)}(\hat{x}^{\pm}, \hat{y}^{\pm}) + \mathcal{O}(1/h),$$
$$\bar{\theta}(\hat{x}^{\pm}, \check{y}^{\pm}) = h\,\bar{\theta}^{(0)}(\hat{x}^{\pm}, \check{y}^{\pm}) + \bar{\theta}^{(1)}(\hat{x}^{\pm}, \check{y}^{\pm}) + \mathcal{O}(1/h),$$

one adds potentials [GV] to finite gap eq. with driving terms. They shift corresponding quasimomenta.

$$\hat{\mathcal{V}}(\hat{x}) = \sum_{j=1}^{\hat{K}_2} \theta^{(1)}(\hat{x}, \hat{x}_{2,j}) + \sum_{j=1}^{\check{K}_2} \bar{\theta}^{(1)}(\hat{x}, \check{x}_{2,j}),$$
$$\check{\mathcal{V}}(\check{x}) = \sum_{j=1}^{\check{K}_2} \theta^{(1)}(\check{x}, \check{x}_{2,j}) + \sum_{j=1}^{\hat{K}_2} \bar{\theta}^{(1)}(\check{x}, \hat{x}_{2,j}).$$

More generally add potentials to all quasimomenta, and consider a one pole excitations $x_n^{\hat{i}\hat{j}}$, $x_n^{\check{i}\check{j}}$. Find the potentials from Bethe equations.

One loop correction to the dressing phases

The full potentials

$$\hat{\mathcal{V}}_k = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \Big(\sum_{\hat{\imath}\hat{\jmath}} (-1)^F \hat{\mathcal{V}}_k^{\hat{\imath}\hat{\jmath}} + \sum_{\check{\imath}\check{\jmath}} (-1)^F \hat{\mathcal{V}}_k^{\check{\imath}\check{\jmath}} \Big).$$

can be written using cot trick:

$$\hat{\mathcal{V}}_k = \frac{1}{4i} \int_{\hat{\mathcal{C}}} dn \cot(\pi n) \sum_{\hat{\imath}\hat{\jmath}} (-1)^F \hat{\mathcal{V}}_k^{\hat{\imath}\hat{\jmath}} + \frac{1}{4i} \int_{\check{\mathcal{C}}} dn \cot(\pi n) \sum_{\check{\imath}\check{\jmath}} (-1)^F \hat{\mathcal{V}}_k^{\check{\imath}\check{\jmath}},$$

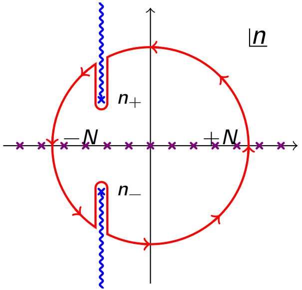
If we look at the BMN frequencies

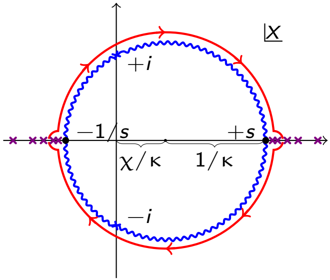
$$\hat{\mathcal{E}}_n = \sqrt{\left(\frac{n}{\mathcal{J}}\right)^2 + 2\chi \frac{n}{\mathcal{J}} + 1} - 1, \qquad \check{\mathcal{E}}_n = \sqrt{\left(\frac{n}{\mathcal{J}}\right)^2 - 2\chi \frac{n}{\mathcal{J}} + 1} - 1.$$

they have branch cuts starting at $n_\pm = \pm i\mathcal{J}(\kappa \pm i\chi)$ running off to infinity

One loop correction to the dressing phases

The contour can be chosen as

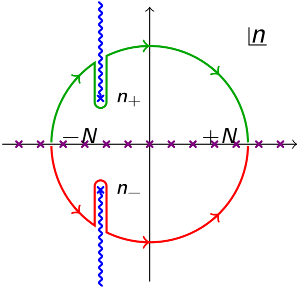


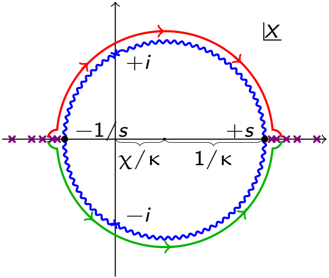


$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n$$

One loop correction to the dressing phases

For large N $\cot(\pi n) \to \mp i$ in the upper/lower half plane.





After cancelation of part of terms

$$\hat{\mathcal{V}}_1^A = +\frac{1}{2}\int_{-s^{-1}}^{+s} \frac{dy}{2\pi}\left((\hat{p}_2^A)' - (\hat{p}_2^S)'\right)\frac{\hat{\alpha}(x)}{x-y} + \frac{1}{2}\int_{-s}^{+s^{-1}} \frac{dy}{2\pi}\left((\check{p}_2^A)' - (\check{p}_2^S)'\right)\frac{\check{\alpha}(1/x)}{\frac{1}{x}-y}$$

One loop correction to the dressing phases

- Express p_2 in terms of resolvents
- Expand integrand at large x
- Perform the integration
- Perform the antisymmetrization

$$\theta^{(1)}(x,y) = -\frac{\hat{\alpha}(x)\hat{\alpha}(y)}{4\pi}\left[\frac{1}{\kappa}\frac{(x+y)\left(1-\frac{1}{xy}\right)-\frac{4\chi}{\kappa}}{(x-s)(x+s^{-1})(y-s)(y+s^{-1})}\frac{x+y}{x-y}\right.$$
$$\left.+\frac{2}{(x-y)^2}\log\left(\frac{y-s}{x-s}\frac{x+s^{-1}}{y+s^{-1}}\right)\right],$$

$$\bar{\theta}^{(1)}(x,y) = -\frac{\hat{\alpha}(x)\check{\alpha}(y)}{4\pi}\left[\frac{1}{\kappa}\frac{(x-y)\left(1+\frac{1}{xy}\right)-\frac{4\chi}{\kappa}}{(x-s)(x+s^{-1})(y+s)(y-s^{-1})}\frac{1+xy}{1-xy}\right.$$
$$\left.+\frac{2}{(1-xy)^2}\log\left(\frac{x+s^{-1}}{x-s}\frac{y-s^{-1}}{y+s}s^2\right)\right].$$

They satisfy $\theta^{(1)}(x,y)+\bar{\theta}^{(1)}(x,1/y) = -\frac{i}{2}\frac{\hat{\alpha}(x)\hat{\alpha}(y)}{(x-y)^2}$ and perfectly match previously known and other results.

Summary

- A set of finite gap equations for string theory on $AdS_3 \times S^3 \times T^4$ with mixed flux was constructed for massive sector, by standard methods reformulating it in formally Z_4 preserving way.
- Using the HT proposed S-matrix with the modified dispersion relation, Bethe equations were written. In thermodynamic limit these equations reproduce the finite gap equations derived from the world-sheet action.
- Few classical string solutions were analysed in finite gap equations framework
- Dressing phases in tree level were conjectured and one loop level correction was derived by one loop quantization of algebraic curve

Outlook

- Full S-matrix derivation from gauge fixed world-sheet *with massless excitations and mixed flux* - done recently
- Bethe equations *with massless excitations and mixed flux* - in progress
- Exact solution for dressing phase from crossing and unitarity relations is needed to proceed with analysis of Bethe equations.
- Generalization to the $AdS_3 \times S^3 \times S^3 \times S^1$ with mixed flux.
- Bethe equations for massless integrable perturbation of corresponding CFT_2 dual would be useful

Thank you