Entanglement entropy in $\mathcal{W}$-algebra CFTs and holography

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Based on :

- Shouvik Datta, Justin R. David, Michael Ferlaino, S. Prem Kumar
  Higher spin entanglement entropy from CFT
  [arXiv:1402.0007] JHEP 1406 (2014) 096.

- Shouvik Datta, Justin R. David, Michael Ferlaino, S. Prem Kumar
  A universal correction to higher spin entanglement entropy
  [arXiv:1405.0015] Phys.Rev.D 90, 041903(R).

- Shouvik Datta
  Relative entropy in higher spin holography
  [arXiv:1406.0520].

Outline

Introduction and motivation

- The study of dualities between theories of higher spin gravity and CFTs with extended symmetries form an important theme in the context of the AdS/CFT correspondence.

  [Klebanov-Polyakov '02, Sezgin-Sundell '02, Gaberdiel-Gopakumar '11]

- Higher spin theories have considerably lesser number of fields as compared to full-fledged string theories.
- At the same time, these theories go beyond classical supergravity and one can hope to capture features of string theory in the tensionless limit.
- CFTs dual to these higher-spin theories are not strongly coupled and are reasonably tractable.
- This enables us to learn a lot from both sides of the duality and understand holography at a deeper level.

Introduction and motivation

- In the holographic context, another remarkable development is the evolution of geometrical methods for calculations of entanglement entropy (EE) in QFTs.
- Entanglement entropy is a useful probe in quantum information theory, QFT and many-body physics.
- Although EE is typically difficult to compute in a QFT – holography offers a simple and elegant route in terms of the Ryu-Takayanagi formula.
- It states that the EE of a subsystem A is given by the minimal surface in AdS that ends on the boundary of A.

  [Ryu-Takayanagi '06]
- A refined observable to study holography.

Questions

- How does the entanglement entropy behave in CFTs with extended symmetries?
- Are there any universal results?

$$S_E = \frac{c}{6} \log \left| \frac{\beta}{\pi\epsilon} \sinh\left(\frac{\pi\Delta}{\beta}\right) \right| + \textbf{?} + \cdots$$

- What is the functional in the bulk higher spin theory which captures the entanglement entropy of its dual CFT?

Entanglement entropy in free CFTs with $\mathcal{W}$-symmetries

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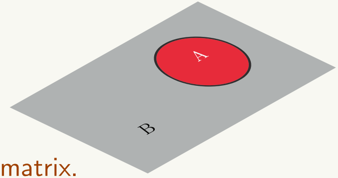
Higher spin entanglement entropy from CFT
[arXiv:1402.0007] JHEP 1406 (2014) 096.

Entanglement and Rényi entropies

- Consider a system (with subsystems A and B) having a Hilbert space which can be written as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- The entanglement entropy is

$$S_E = -\mathrm{tr}_A\, \rho_A \log \rho_A$$



where $\rho_A = \mathrm{tr}_B(\rho)$ is the reduced density matrix.
[Bombelli-Koul-Lee-Sorkin '86]

- Rényi entropies are defined as

$$S_n = \frac{1}{1-n} \log \mathrm{tr}_A\, \rho_A^n$$

It follows that $\lim_{n\to 1} S_n = S_E$.

Entanglement and Rényi entropies

An example : A system of 2 spins

Consider the state $|\Psi\rangle = \cos\theta\,|\uparrow\downarrow\rangle + \sin\theta\,|\downarrow\uparrow\rangle$

Density matrix : $\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta & \cos\theta\sin\theta & 0 \\ 0 & \cos\theta\sin\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

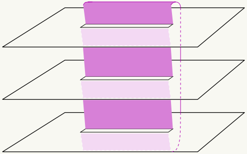
Reduced density matrix : $\rho_A = \mathrm{tr}_B(\rho) = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

For $|\mathsf{EPR}\rangle = \frac{1}{\sqrt{2}}\,|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}\,|\downarrow\uparrow\rangle$ we have $\theta = \pi/4$ and $S_A = \log 2$.

Entanglement and Rényi entropies

Replica trick : Computing $\mathrm{tr}\,\rho_A^n$ by making n copies of the system and then sewing them together cyclically along the cuts which define the sub-system A.

[Hozley-Larsen-Wilczek '94; Cardy-Calabrese '04]



If the partition function on this orbifold or branched Riemann surface ($\mathcal{R}_n = \mathbb{C}/\Gamma$) is denoted by Z_n then

$$\mathrm{tr}\,\rho_A^n = \frac{Z_n(A)}{Z_1^n} \qquad S_n = \frac{1}{1-n}\log\frac{Z_n}{Z_1^n}$$

One can then find the Rényi entropy S_n and then analytically continue it to $n \to 1$ to obtain the entanglement entropy S_E.

Rényi entropies and twist operators

Finding Z_n amounts to finding the partition function of n-copies of the QFT which has fields obeying certain cyclic conditions

$$\varphi_i(x, 0^+) = \varphi_{i+1}(x, 0^-) \text{ for } x \in [u, v]$$

These can be implemented what are known as twist operators.

The twist operators are generators of cyclic permutation symmetry of the orbifold.

The partition function on the orbifolded/branched surface can then be written as a correlation function of twist and anti-twist operators.

$$Z_n \propto \langle \mathcal{T}(u, 0)\bar{\mathcal{T}}(v, 0)\rangle$$

[Dixon-Friedan-Matinec-Shenker '87; Atick-Sen '87; Atick-Dixon-Griffin-Nemechansky '88; Saleur '88;..]

Free fermion and free boson CFTs

- We shall now consider the free fermion and free boson CFTs in 1+1 dimensions at finite temperature (β) – on the infinite cylinder $\mathbb{R} \times S^1_\beta$.
- These CFTs have an infinite set of conserved currents of increasing spin.
- We shall deform the action by turning on a chemical potential (μ) corresponding to a spin-3 current.

$$Z = \int \mathcal{D}\phi \exp\Big[-\int d^2 z \big(\mathcal{L} - \mu(W(z) + W(\bar{z}))\big)\Big]$$

[Dijkgraaf '96, deBoer-Jottar '14]

- We shall then evaluate corrections to the known universal formula for EE perturbatively in μ/β.

Conformal perturbation theory

Thermal entropy

The partition function of the deformed theory is given in holomorphic conformal perturbation theory as

$$Z = Z^{(0)}_{\rm CFT}\left[1-\mu\int d^2z\,\langle W(z)\rangle_{\rm CFT}+\frac{\mu^2}{2}\int d^2z_1\int d^2z_2\,\langle W(z_1)W(z_2)\rangle_{\rm CFT}+\cdots\right]$$

The finite temperature correlators can be obtained from those on the plane by conformal transformation – $z=\frac{\beta}{2\pi}\log w$.

$$\big\langle W(z_1)W(z_2)\big\rangle = \mathcal{N}\frac{\pi^6}{\beta^6\sinh^6\frac{\pi}{\beta}(z_1-z_2)}$$

After performing the integrals, one gets

$$\frac{\log Z}{L}=\frac{\pi c}{6\beta}+\frac{8\pi^3 c}{9\beta^3}\mu^2+\cdots$$

which is a corrected Cardy's formula in presence of a spin-3 chemical potential. [Gaberdiel-Hartman-Jin '12; SD-David-Ferlaino-Kumar '13; Long '14]

Conformal perturbation theory

Entanglement and Rényi entropies

The partition function Z_n on $\mathcal{R}_n =$ correlation function of twist fields.

$$Z_n = \prod_k \left\langle \sigma_{k,n}(y_1, \bar{y}_1)\bar{\sigma}_{k,n}(y_2, \bar{y}_2)\right\rangle$$

When the deformation is turned on, this becomes

$$Z_n = \prod_k \left\langle \sigma_{k,n}(y_1, \bar{y}_1)\bar{\sigma}_{k,n}(y_2, \bar{y}_2)\, e^{-\mu \int d^2z\, W(z)}\right\rangle$$

k is a label for the eigenfunctions and eigenvalues ($e^{2\pi i k/n}$) of the twist operator.
Bosons : $k = 0, 1, \cdots, n-1$. Fermions : $k = -\frac{n-1}{2}, \cdots, \frac{n-1}{2}$.

Conformal perturbation theory

Entanglement and Rényi entropies

The partition function Z_n on $\mathcal{R}_n$ = correlation function of twist fields.

$$Z_n = \prod_k \left\langle \sigma_{k,n}(y_1,\bar{y}_1)\bar{\sigma}_{k,n}(y_2,\bar{y}_2)\right\rangle$$

When the deformation is turned on, this becomes

$$Z_n = \prod_k \Big(\left\langle \sigma_{k,n}(1)\bar{\sigma}_{k,n}(2)\right\rangle - \mu \int d^2z \left\langle \sigma_{k,n}(1)W(z)\bar{\sigma}_{k,n}(2)\right\rangle + \frac{\mu^2}{2}\int d^2z_1 \int d^2z_2 \left\langle \sigma_{k,n}(1)W(z_1)W(z_2)\bar{\sigma}_{k,n}(2)\right\rangle + \cdots \Big)$$

We thus need to find correlation functions of twist-fields with insertions of the spin-3 operators.

k is a label for the eigenfunctions and eigenvalues ($e^{2\pi i k/n}$) of the twist operator. Bosons : $k = 0, 1, \cdots, n-1$. Fermions : $k = -\frac{n-1}{2}, \cdots, \frac{n-1}{2}$.

Free fermions

- Consider N free massless complex fermions, $\mathcal{L} = \bar{\psi}_a \partial\!\!\!/ \psi_a$.
- This theory has conserved currents of $s = 1, 2, 3, \cdots$ the modes of which form a $\mathcal{W}_{1+\infty}$ algebra.

  [Pope-Romans-Shen '90, Bergshoeff-Pope-Romans-Sezgin-Shen '90]

Bosonization

We shall work in the bosonized language in which the twist operators have an explicit field representation.

$$\psi^{k,a} = : e^{i\varphi} : \qquad \bar{\psi}^{k,a} = : e^{-i\varphi} :$$

The twist operators are $\sigma(z, \bar{z}) = \prod_a : e^{i\frac{k}{n}\left[\varphi_{a,k} - \bar{\varphi}_{a,k}\right]} :$

The spin-3 current is $W = -\frac{\sqrt{5}}{6\pi} \sum_a : (\partial\varphi_a)^3 :$
It is a (3,0) primary in the $J = 0$ sector.

Free fermions

By Wick contractions, one can find the correlators $\langle \sigma(1) W(z) \bar{\sigma}(2) \rangle$ and $\langle \sigma(1) W(z_1) W(z_2) \bar{\sigma}(2) \rangle$. We can then conformal transform them to the cylinder $z_{\text{plane}} = e^{2\pi w_{\text{cyl}}/\beta}$.

After performing the integrals, we have the following expressions.

Rényi entropies

$$S_n(\Delta) = \frac{N(n+1)}{6n} \log \left| \frac{\beta}{\pi} \sinh\left(\frac{\pi\Delta}{\beta}\right) \right| + \frac{5N\mu^2}{6\pi^2} \left[\frac{1+n}{4n} \mathcal{I}_1(\Delta) - \frac{(1+n)(7-3n^2)}{160n^3} \mathcal{I}_2(\Delta) \right]$$

Entanglement entropy

$$S_E(\Delta) = \frac{N}{3} \log \left| \frac{\beta}{\pi} \sinh\left(\frac{\pi\Delta}{\beta}\right) \right| + \frac{5N\mu^2}{6\pi^2} \left[\frac{1}{2} \mathcal{I}_1(\Delta) - \frac{1}{20} \mathcal{I}_2(\Delta) \right] + \mathcal{O}(\mu^3)$$

These reduce to the thermal entropy in the extensive limit, $\Delta \gg \beta$.

Integrals

The expressions for the integrals $\mathcal{I}_1(\Delta, \beta)$ and $\mathcal{I}_2(\Delta, \beta)$ are as follows

$$\mathcal{I}_1\,(\Delta) \;=\; \int d^2 z_1 \int d^2 z_2 \;\, H^4(z_1 - z_2)\, G(z_1)\, G(z_2)$$

$$\mathcal{I}_2\,(\Delta) \;=\; \int d^2 z_1 \int d^2 z_2 \;\, H^2(z_1 - z_2)\, G^2(z_1)\, G^2(z_2)$$

where

$$H(z) \;\equiv\; \frac{\pi}{\beta \sinh\left(\frac{\pi\, z}{\beta}\right)}\,, \qquad\qquad G(z) \;\equiv\; \frac{\pi \sinh\left(\frac{\pi\,\Delta}{\beta}\right)}{\beta \sinh\left(\frac{\pi\, z}{\beta}\right)\, \sinh\left(\frac{\pi\,(z-\Delta)}{\beta}\right)}\,.$$

$$\mathcal{I}_1 = \tfrac{4\pi^4}{3\beta^2}\, \left(\tfrac{4\pi\Delta}{\beta}\, \coth\left(\tfrac{\pi\Delta}{\beta}\right) - 1\right) + \frac{4\pi^4}{\beta^2 \sinh^2\left(\frac{\pi\Delta}{\beta}\right)}\, \left\{\left(1 - \tfrac{\pi\Delta}{\beta} \coth\left(\tfrac{\pi\Delta}{\beta}\right)\right)^2 - \left(\tfrac{\pi\Delta}{\beta}\right)^2\right\}$$

$$\mathcal{I}_2 = \tfrac{8\pi^4}{\beta^2}\, \left(5 - \tfrac{4\pi\Delta}{\beta}\, \coth\left(\tfrac{\pi\Delta}{\beta}\right)\right) + \frac{72\pi^4}{\beta^2 \sinh^2\left(\frac{\pi\Delta}{\beta}\right)}\, \left\{\left(1 - \tfrac{\pi\Delta}{\beta} \coth\left(\tfrac{\pi\Delta}{\beta}\right)\right)^2 - \tfrac{1}{9}\, \left(\tfrac{\pi\Delta}{\beta}\right)^2\right\}$$

Free bosons

- Consider N free massless complex bosons, $\mathcal{L} = \bar{\partial}\bar{X}_a \partial X_a + \partial \bar{X}_a \bar{\partial} X_a$.
- This theory has conserved currents of $s = 2, 3, \cdots$ the modes of which form a $\mathcal{W}_\infty[1]$ algebra.
  [Bakas-Kiritsis '90]

$$T(z) = -\partial \bar{X}_a \partial X_a \qquad W(z) = \sqrt{\frac{5}{12\pi^2}}(\partial^2 \bar{X}_a \partial X_a - \partial \bar{X}_a \partial^2 X_a)$$

Twist operators?

The twist operators cannot be written explicitly in terms of free fields.

They are abstractly known in terms of OPEs only.

[Dixon-Friedan-Martinec-Shenker '87]

Free bosons

We need to find the correlators $\langle\sigma(1)W(z)\bar{\sigma}(2)\rangle$ and $\langle\sigma(1)W(z_1)W(z_2)\bar{\sigma}(2)\rangle$.

Consider the following Green's functions having insertions of pairs of $\partial_z\bar{X}(z)\partial_w X(w)$

$$g(z,w;y_1) = \frac{\langle -\partial_z\bar{X}(z)\partial_w X(w)\bar{\sigma}_{k,n}(1)\sigma_{k,n}(2)\rangle}{\langle\bar{\sigma}_{k,n}(1)\sigma_{k,n}(2)\rangle}$$

$$f(z,w,z',w';y_1) = \frac{\langle -\partial_z\bar{X}(z)\partial_w X(w)\partial_{z'}\bar{X}(z')\partial_{w'}X(w')\bar{\sigma}_{k,n}(1)\sigma_{k,n}(2)\rangle}{\langle\bar{\sigma}_{k,n}(1)\sigma_{k,n}(2)\rangle}$$

Investigating behaviours as $z \to w, y_i$ and $w \to y_i$, the functions can be fixed completely.

We then need to take appropriate combinations of derivatives of these Green's functions to get the required correlators.

Free bosons

After performing the integrals, one has

Rényi entropies

$$S_n(\Delta) = \frac{c(n+1)}{6n}\log\left|\frac{\beta}{\pi}\sinh\left(\frac{\pi\Delta}{\beta}\right)\right| + \frac{5c\mu^2}{6\pi^2}\left[\frac{1+n}{4n}\mathcal{I}_1(\Delta) - \frac{(n+1)(n^2-4)}{120n^3}\mathcal{I}_2(\Delta)\right]$$

Entanglement entropy

$$S_E(\Delta) = \frac{c}{3}\log\left|\frac{\beta}{\pi}\sinh\left(\frac{\pi\Delta}{\beta}\right)\right| + \frac{5c\mu^2}{6\pi^2}\left[\frac{1}{2}\mathcal{I}_1(\Delta) - \frac{1}{20}\mathcal{I}_2(\Delta)\right] + \mathcal{O}(\mu^3)$$

Free bosons

After performing the integrals, one has

Rényi entropies

$$S_n(\Delta) = \frac{c(n+1)}{6n}\log\left|\frac{\beta}{\pi}\sinh\left(\frac{\pi\Delta}{\beta}\right)\right| + \frac{5c\mu^2}{6\pi^2}\left[\frac{1+n}{4n}\mathcal{I}_1(\Delta) - \frac{(n+1)(n^2-4)}{120n^3}\mathcal{I}_2(\Delta)\right]$$

Entanglement entropy

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The expression for the correction to entanglement entropy matches exactly with that of free fermions.

Entanglement entropy in higher spin holography

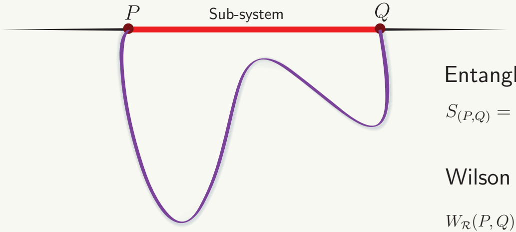
Higher spin gravity in 3d and entanglement entropy

- It has emerged recently that CFTs with $\mathcal{W}$-symmetry are dual to higher spin theories of gravity in AdS_3.
  [Gaberdiel-Gopakumar '10]
- Higher spin gravity in 3 dimensions admits a simple description in terms of a Chern-Simons theory based on a gauge group ($sl(N,\mathbb{R})$, $\mathrm{hs}[\lambda]$ etc.)
  [Blencowe '89; Prokushkin-Vasiliev '98, '99]
- Since these theories go beyond diffeomorphism invariance, one needs to rethink geometrical notions of black hole horizons and minimal surfaces.
- There exist explicit constructions of black holes and conical defects in this theory which are characterized in terms of holonomies.
- What is the quantity that generalizes the notion of the Ryu-Takayanagi minimal surfaces in a higher spin theory?

Higher spin gravity in 3d and entanglement entropy

It has been proposed that bulk observable which captures EE is terms of a Wilson line anchored at the endpoints of the entangling interval.

[Ammon-Castro-Iqbal '13; deBoer-Jottar '13; Castro-Llabrés '14]



Entanglement entropy

$$S_{(P,Q)} = \frac{\mathsf{c}}{\sigma_{\mathcal{R}}} \log \left[\lim_{\rho_0 \to \infty} W_{\mathcal{R}}(P,Q) \Big|_{\rho_P = \rho_Q = \rho_0} \right]$$

Wilson line functional in the bulk

$$W_{\mathcal{R}}(P,Q) \equiv \mathrm{tr}_{\mathcal{R}} \left[\mathcal{P} \exp \left(\int_P^Q \bar{A}_{\bar{z}} \right) \mathcal{P} \exp \left(\int_Q^P A_z \right) \right]$$

Higher spin gravity in 3d and entanglement entropy

- The bulk configuration at finite temperature and carrying a higher-spin charge is that of the higher spin black hole.
  [Gutperle-Kraus '11]
- We shall consider black holes in the simplest higher spin theory – based on the gauge group $sl(3,\mathbb{R})$.
- Confining our attention to the BTZ-branch, the EE is computed via the Wilson line functional to be

$$S_E = \frac{c}{3}\log\left|\frac{\pi}{\beta}\sinh\left(\frac{\pi\Delta}{\beta}\right)\right| + c\,\frac{\mu^2}{\beta^2}\left[\frac{32\pi^2}{9}\left(\frac{\pi\Delta}{\beta}\right)\coth\left(\frac{\pi\Delta}{\beta}\right) - \frac{20\pi^2}{9}\right.$$
$$\left. - \frac{4\pi^2}{3}\mathrm{csch}^2\left(\frac{\pi\Delta}{\beta}\right)\left\{\left(\frac{\pi\Delta}{\beta}\coth\left(\frac{\pi\Delta}{\beta}\right) - 1\right)^2 + \left(\frac{\pi\Delta}{\beta}\right)^2\right\}\right] + \mathcal{O}(\mu^4)$$

.
.

Higher spin gravity in 3d and entanglement entropy

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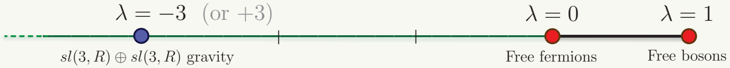
$$\left. - \frac{4\pi^2}{3}\text{csch}^2\left(\tfrac{\pi\Delta}{\beta}\right)\left\{\left(\tfrac{\pi\Delta}{\beta}\coth\left(\tfrac{\pi\Delta}{\beta}\right)-1\right)^2 + \left(\tfrac{\pi\Delta}{\beta}\right)^2\right\}\right] + \mathcal{O}(\mu^4)$$

The first correction to the entanglement entropy is exactly the same as that of the free boson and free fermion theories!

Universality – a posteriori

Quite surprisingly, the entanglement entropy to $\mathcal{O}(\mu^2)$ is exactly the same for two distinct free field theories and also from a holographic calculation.



Space of $\mathcal{W}_\infty[\lambda]$ theories

Is this suggestive of an universal correction to higher spin entanglement entropy?

Universality of higher-spin corrections to entanglement entropy

Shouvik Datta, Justin R. David, Michael Ferlaino, S. Prem Kumar
A universal correction to higher spin entanglement entropy
[arXiv:1405.0015] Phys.Rev.D 90, 041903(R).

Correlators on the n-sheeted Riemann surface

- We had just seen that the calculation of the higher spin correction to EE is tantamount to computing one- and two-point functions 'in the twisted sector' – $\langle \bar{\sigma}(y_1) W(z) \sigma(y_2) \rangle$ and $\langle \bar{\sigma}(y_1) W(z_1) W(z_2) \sigma(y_2) \rangle$.
- This is equivalent to finding the correlators on the replica geometry which is a n-sheeted Riemann surface $\mathcal{R}_n$

$$\langle W(z) \rangle_{\mathcal{R}_n} = \frac{\langle 0|\bar{\sigma}(y_1) W(z) \sigma(y_2)|0\rangle}{\langle 0|\bar{\sigma}(y_1) \sigma(y_2)|0\rangle}$$

$$\langle W(z_1) W(z_2) \rangle_{\mathcal{R}_n} = \frac{\langle 0|\bar{\sigma}(y_1) W(z_1) W(z_2) \sigma(y_2)|0\rangle}{\langle 0|\bar{\sigma}(y_1) \sigma(y_2)|0\rangle}$$

- Can we try to find such multi-sheeted correlators in the $\mathcal{W}_\infty[\lambda]$ theory $\forall \lambda$?

Correlators on the n-sheeted Riemann surface

By conformal invariance the required correlator can be expressed as

$$\langle \bar{\sigma}(y_1, \bar{y}_1) W(z_1) W(z_2) \sigma(y_2, \bar{y}_2) \rangle = -\frac{5c/6\pi^2}{(z_{12})^6} \frac{1}{|y_{12}|^{2d_n}} F(x)$$

Here $x = \frac{(z_1 - y_2)(z_2 - y_1)}{(z_1 - y_1)(z_2 - y_2)}$, which is the conformal cross-ratio.

$d_n = \frac{c}{12}(n - \frac{1}{n})$ is the dimension of the twist operator.

Properties of $F(x)$

- Invariance under exchanges $z_1 \leftrightarrow z_2$ and $y_1 \leftrightarrow y_2$: $F(x) = F(1/x)$.
- $W(z_1)W(z_2)$ OPE $z_1 \to z_2$: $F(x \to 1) = 1$.
- W approaching a twist $z_1 \to y_{1,2}$ or $z_2 \to y_{1,2}$: $F(x \to \infty) \sim x^M$ and $F(x \to 0) \sim x^{-M}$. (next slide)

Correlators on the n-sheeted Riemann surface

The value of M is constrained by the OPE of the spin-3 current with the twist

$$W(z)\sigma_n(y) \sim \frac{1}{(z-y)^M}\sigma_n^{(\mathrm{ex})}(y) + \cdots$$

The excited twists have dimension greater than d_n. Thus, $M < 3$ by dimensional analysis.

Also, there shouldn't be any branch cuts because $W = \sum_i W_i$ is invariant under the action of twists. This forces $M = 2$.

These requirements specify $F(x)$ to be

$$F(x) \equiv F(\eta) = 1 + f_1\eta + f_2\eta^2$$

η is the symmetrized cross ratio given by

$$\eta \equiv x + \frac{1}{x} - 2 = \frac{(z_1-z_2)^2(y_1-y_2)^2}{(z_1-y_1)(z_1-y_2)(z_2-y_1)(z_2-y_1)}$$

Correlators and $\mathcal{W}_\infty[\lambda]$ OPEs

The unknown coefficients $f_{1,2}$ can be determined by comparing the behaviour of the 4-point correlator $\left\langle \bar{\sigma}(y_1, \bar{y}_1) W(z_1) W(z_2) \sigma(y_2, \bar{y}_2) \right\rangle$ in the $z_1 \to z_2$ limit with OPEs of the $\mathcal{W}_\infty[\lambda]$ algebra.

$$
\begin{aligned}
\frac{1}{N_3} W(z_1) W(z_2) \sim & \frac{5c/6}{(z_1 - z_2)^6} + \frac{5T(z_2)}{(z_1 - z_2)^4} + \frac{5T'(z_2)/2}{(z_1 - z_2)^3} \\
& + \frac{\frac{4}{N_3} U(z_2) + \frac{16}{c+22/5} \Lambda^{(4)}(z_2) + \frac{3}{4} T''(z_2)}{(z_1 - z_2)^2} \\
& + \frac{\frac{2}{N_3} \partial U(z_2) + \frac{8}{c+22/5} \partial \Lambda^{(4)}(z_2) + \frac{1}{6} T'''(z_2)}{z_1 - z_2}
\end{aligned}
$$

We need to find the expectation values of the operators on $\mathcal{R}_n$ appearing on the OPE above.

Correlators and $\mathcal{W}_\infty[\lambda]$ OPEs

The unknown coefficients $f_{1,2}$ can be determined by comparing the behaviour of the 4-point correlator $\langle \bar{\sigma}(y_1, \bar{y}_1) W(z_1) W(z_2) \sigma(y_2, \bar{y}_2) \rangle$ in the $z_1 \to z_2$ limit with OPEs of the $\mathcal{W}_\infty[\lambda]$ algebra.

$$\frac{1}{N_3}\langle W(z_1)W(z_2)\rangle \sim \frac{5c/6}{(z_1-z_2)^6} + \frac{5\langle T(z_2)\rangle}{(z_1-z_2)^4} + \frac{5\langle T'(z_2)\rangle/2}{(z_1-z_2)^3} + \frac{\frac{4}{N_3}\langle U(z_2)\rangle + \frac{16}{c+22/5}\langle \Lambda^{(4)}(z_2)\rangle + \frac{3}{4}\langle T''(z_2)\rangle}{(z_1-z_2)^2} + \frac{\frac{2}{N_3}\partial\langle U(z_2)\rangle + \frac{8}{c+22/5}\partial\langle \Lambda^{(4)}(z_2)\rangle + \frac{1}{6}\langle T'''(z_2)\rangle}{z_1-z_2}$$

We need to find the expectation values of the operators on $\mathcal{R}_n$ appearing on the OPE above.

The uniformization transformation

There exists a uniformization transformation which maps the entire n-sheeted Riemann surface $\mathcal{R}_n$ to the complex plane $\mathbb{C}$.

$$w_{\text{plane}} = \left(\frac{z - y_2}{z - y_1}\right)^{1/n}$$

[Cardy-Calabrese '09]

The one-point functions of the operators on $\mathcal{R}_n$ which appear in the OPEs can be found by using the conformal transformation above.

Example : The one-point function of the stress tensor.

$$T(z) = w'(z)^2 T(w) + \frac{c}{12}\{w, z\}$$

The non-vanishing contribution is only from the Schwarzian derivative.

$$\langle T(z) \rangle_{\mathcal{R}_n} = \frac{c}{24}\frac{(n^2-1)}{n^2}\frac{\Delta^2}{(z-y_1)^2(z-y_2)^2}$$

Other composite operators $\langle \Lambda^{(4)} \rangle = \langle\ :TT: - \frac{3}{10}\partial^2 T \rangle$ can also be found by point-splittings.

Fixing the correlator

We can fix the correlator $\langle\bar{\sigma}(y_1,\bar{y}_1)W(z_1)W(z_2)\sigma(y_2,\bar{y}_2)\rangle$ by comparing

$$\frac{1}{N_3}\langle W(z_1)W(z_2)\rangle \sim \frac{5c/6}{(z_1-z_2)^6}+\frac{5\langle T(z_2)\rangle}{(z_1-z_2)^4}+\frac{5\langle T'(z_2)\rangle/2}{(z_1-z_2)^3}$$

$$+\frac{\cancel{\frac{4}{N_3}\langle U(z_2)\rangle}+\frac{16}{c+22/5}\langle\Lambda^{(4)}(z_2)\rangle+\frac{3}{4}\langle T''(z_2)\rangle}{(z_1-z_2)^2}$$

Just the $\mathcal{W}_3$ subalgebra contributes!

$$+\frac{\cancel{\frac{2}{N_3}\langle\partial U(z_2)\rangle}+\frac{8}{c+22/5}\langle\partial\Lambda^{(4)}(z_2)\rangle+\frac{1}{6}\langle T'''(z_2)\rangle}{z_1-z_2}$$

with

the Laurent series expansion in (z_1-z_2) of

$$\langle W(z_1)W(z_2)\rangle=\frac{\langle\bar{\sigma}(y_1,\bar{y}_1)W(z_1)W(z_2)\sigma(y_2,\bar{y}_2)\rangle}{\langle\bar{\sigma}(y_1,\bar{y}_1)\sigma(y_2,\bar{y}_2)\rangle}=-\frac{5c/6\pi^2}{(z_{12})^6}(1+f_1\eta+f_2\eta^2)$$

Fixing the correlator

We obtain

$$\langle W(z_1)W(z_2)\rangle_{\mathcal{R}_n} = \frac{\langle \bar{\sigma}(y_1,\bar{y}_1)W(z_1)W(z_2)\sigma(y_2,\bar{y}_2)\rangle}{\langle \bar{\sigma}(y_1,\bar{y}_1)\sigma(y_2,\bar{y}_2)\rangle} = -\frac{5c/6\pi^2}{(z_{12})^6}(1+f_1\eta+f_2\eta^2)$$

with

$$f_1 = \frac{n^2-1}{4n^2} \qquad f_2 = \frac{(n^2-1)^2}{120n^4} - \frac{(n^2-1)}{40n^4}$$

This is true for a CFT with a $\mathcal{W}_\infty[\lambda]$ symmetry for any λ.
(Also, matches with the correlator for the free boson CFT calculated earlier.)

One can also perform the same exercise using the OPEs of the $\mathcal{W}_{1+\infty}$ CFT and see that it does match with that of the free fermions.

Consistency check : This method of evaluating $\langle T(z_1)T(z_2)\rangle_{\mathcal{R}_n}$ agrees with the expression for the same calculated from Ward identities.

Universality of the correlation function

- We have proved that the correlator $\langle W(z_1)W(z_2)\rangle$ on the n-sheeted Riemann surface $\mathcal{R}_n$ is universal for a $\mathcal{W}_\infty[\lambda]$ CFT.
- In order to get finite temperature correlators, we conformal transform to the cylinder $\mathbb{R} \times S^1_\beta$.
- One needs to compute the integrals of this two point function to obtain the correction to the partition function on the replica geometry.
- This implies that the $\mathcal{O}(\mu^2)$ correction to the entanglement/Renyi entropy is universal.
- The results are true for all values of the central charge.
- The methods employed here can be used more generally to study deformations of CFTs by holomorphic operators.

Relative entropy in higher-spin holography

Shouvik Datta
Relative entropy in higher spin holography
[arXiv:1406.0520].

Relative entropy

Definition and properties

- Relative entropy is a measure of distinguishability of two states for a quantum system.

- For two density matrices σ and ρ, the relative entropy is defined as

$$\mathcal{S}(\sigma||\rho) = \mathrm{tr}(\sigma \log \sigma) - \mathrm{tr}(\sigma \log \rho)$$

- Properties
  1. Non-negativity : $\mathcal{S}(\sigma||\rho) \geq 0$.
  2. Invariance under unitary trans : $\mathcal{S}(\sigma||\rho) = \mathcal{S}(U^\dagger \sigma U || U^\dagger \rho U)$.
  3. Monotonicity under partial traces : $\mathcal{S}(\sigma||\rho) \geq \mathcal{S}(\mathrm{tr}_P \sigma || \mathrm{tr}_P \rho)$
  4. Additivity : $\mathcal{S}(\sigma_A \otimes \sigma_B||\rho) = \mathcal{S}(\sigma_A||\rho) + \mathcal{S}(\sigma_B||\rho)$

[Vedral '02]

Relative entropy

Relationship with the modular Hamiltonian and entanglement entropy

For a given (reduced) density matrix, the modular Hamiltonian is defined as

$$\rho = \frac{e^{-H}}{\mathrm{tr}(e^{-H})}$$

It can then be shown that the relative entropy is

$$\mathcal{S}(\sigma||\rho) = \Delta\langle H\rangle - \Delta S$$

The relative entropy vanishes in the limit of small sub-system sizes

$$\lim_{\frac{\mathrm{dim}(A)}{\mathrm{dim}(A')}\to 0}\left(\Delta\langle H_A\rangle - \Delta S_A\right) = 0 \qquad \Longrightarrow \qquad \Delta\langle H\rangle = \Delta S$$

The first law of entanglement

[Blanco-Casini-Hung-Myers '13]

Relative entropy in a $\mathcal{W}$-algebra CFT and its holographic dual

- We shall try to calculate the relative entropy between a high temperature state and the vacuum in a CFT with $\mathcal{W}$ symmetries in presence of a chemical potential for the spin-3 current.

  The CFT is at large central charge and on a finite system of size R and the high temperature state is at temperature T.

- As we had seen earlier such a CFT is describable in terms of higher-spin gravity.

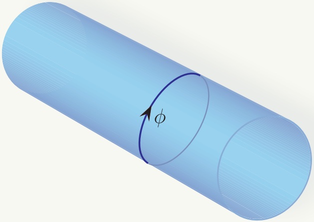
- It is possible to calculate $\langle H_A \rangle$ from the holographic stress tensor. The EE (S_A) is also calculable in terms of Wilson lines.

- We shall try to verify $\Delta\langle H_A \rangle = \Delta S$ in the short distance regime.

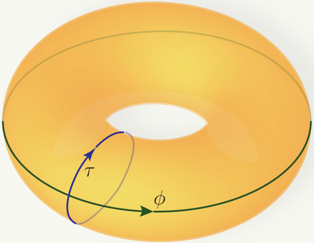
The bulk configurations

The gravity configurations dual to the vacuum and high temperature state of the CFT are the higher spin vacuum and black hole respectively.

[Gutperle-Kraus ‘11; Kraus-Perlmutter ‘11; Castro-Gopakumar-Gutperle-Raeymaekers ‘11; Li-Lin-Wang ‘13; Compere-Jottar-Song ‘13; Chowdhury-Saha ‘13]



Vacuum



High temperature state

The higher spin vacuum is a higher spin generalization of global AdS. It has trivial holonomy along the spatial ϕ cycle.

The higher spin black hole generalizes the BTZ. Its temporal cycle τ has trivial holonomy.

Modular Hamiltonians in CFT

The modular Hamiltonian is not a local quantity in general. However, there exist special cases where it is local and calculable.

[Casini-Huerta-Myers '11]

The modular Hamiltonian associated with the vacuum in a 1+1 d CFT is

$$H_{\mathsf{vac}} = 2\pi R^2 \int_{-\frac{\phi}{2}}^{\frac{\phi}{2}} d\theta \, \frac{\cos\theta - \cos\frac{\phi}{2}}{\sin\frac{\phi}{2}} \, T_{00}(\theta)$$

Here, $T_{00} = (L_0 - \frac{c}{24}) + (\bar{L}_0 - \frac{c}{24})$. These can be obtained from the holographic stress tensor for specific states.

[Balasubramanian-Kraus '99, de Haro-Solodukhin-Skenderis '00]

Modular Hamiltonian from holographic stress tensor

The stress tensors corresponding to the hs-vacuum and the hs-black hole can be obtained by solving holonomy conditions.

When a higher spin chemical potential is turned on perturbatively, the $\mathcal{W}_3 \times \mathcal{W}_3$ asymptotic symmetry is unbroken.

[Compere-Song '13; Compere-Jottar-Song '13; deBoer-Jottar '14]

The expectation values of the modular Hamiltonian are therefore

$$\langle H \rangle_{\text{state}} = \text{tr}(\rho_{\text{state}} H_{\text{vac}}) = 8\pi R^2 \left[1 - \frac{\phi}{2}\cot\left(\frac{\phi}{2}\right)\right] \mathcal{L}_{\text{state}}$$

where,

$$\mathcal{L}_T = \frac{c\pi T^2}{12}\left[1 + \frac{80(\pi\mu T)^2}{3} + \frac{2560(\pi\mu T)^4}{3} + \frac{905216(\pi\mu T)^6}{27} + \cdots\right]$$

$$\mathcal{L}_{\text{vac}} = -\frac{c}{48\pi R^2}\left[1 - \frac{20}{3}\left(\frac{\mu}{R}\right)^2 + \frac{160}{3}\left(\frac{\mu}{R}\right)^4 - \frac{14144}{27}\left(\frac{\mu}{R}\right)^6 + \cdots\right]$$

The difference $\Delta\langle H \rangle$ can then be calculated.

Holographic entanglement entropy

The EEs – computed via Wilson lines – corresponding to higher spin black holes and the vacuum in the $sl(3)$ theory are

$$
\begin{aligned}
S_T(\phi) =& \frac{c}{3}\log\left|\frac{\sinh(\pi R T\phi)}{\Lambda^{-1}\,\pi T}\right| \\
&+\frac{c}{18}(\pi\mu T)^2 \mathsf{csch}^4(\pi R T\phi)\,\Big[8\left(1-3\pi^2R^2T^2\phi^2\right)\cosh(2\pi R T\phi) \\
&\qquad +8\pi R T\phi\left(\sinh(2\pi R T\phi)+\sinh(4\pi R T\phi)\right) \\
&\qquad -5\cosh(4\pi R T\phi)-3\,\Big]+\mathcal{O}((\pi\mu T)^4)
\end{aligned}
$$

$$
\begin{aligned}
S_{\mathsf{vac}}(\phi) =& \frac{c}{3}\log\left|\frac{2R}{\Lambda^{-1}}\,\sin\left(\frac{\phi}{2}\right)\right| \\
&+\frac{c}{72}\left(\frac{\mu}{R}\right)^2\csc^4\left(\frac{\phi}{2}\right)\Big[3-2\left(3\phi^2+4\right)\cos(\phi)+4\phi(\sin(\phi)+\sin(2\phi)) \\
&\qquad +5\cos(2\phi)\Big]+\mathcal{O}((\mu/R)^4)
\end{aligned}
$$

One can systematically keep track of terms to higher orders.

Relative entropy in holographic CFTs with a $\mathcal{W}$-symmetry

We can now employ the thermodynamic-like relation to calculate the relative entropy between the high-temp state and the vacuum.

$$\mathcal{S}(\rho_T||\rho_{\mathsf{vac}}) = (\langle H \rangle_T - \langle H \rangle_{\mathsf{vac}}) - (S_T - S_{\mathsf{vac}})$$

We shall focus on the small-subsystem size regime where we expect $\Delta\langle H \rangle = \Delta S$.

Relative entropy in holographic CFTs with a $\mathcal{W}$-symmetry

The following can then be established via holographic computations.

$$\Delta S\Big|_{\text{to } \phi^2} = \Delta H\Big|_{\text{to } \phi^2}$$

$$= c\,\phi^2\left[\frac{((\ell T)^2+1)}{72} + \frac{5\left((\ell T)^4-1\right)}{54}\frac{\mu^2}{R^2} + \frac{20\left((\ell T)^6+1\right)}{27}\frac{\mu^4}{R^4} + \frac{1768\left((\ell T)^8-1\right)}{243}\frac{\mu^6}{R^6} + \frac{57664\left((\ell T)^{10}+1\right)}{729}\frac{\mu^8}{R^8} + \ldots\right]$$

At the leading order in entangling interval sizes, $\Delta H = \Delta S$ in a large-c CFT with a $\mathcal{W}_3$ symmetry at finite higher spin chemical potential. ($\ell = 2\pi R$)

If the AdS is considered as the ultimate vacuum, $\Delta\langle H\rangle = \Delta S$ can be verified for that case as well.

Relative entropy in holographic CFTs with a $\mathcal{W}$-symmetry

Comments

- We have verified the first law of entanglement holographically – in the regime of short intervals and at finite chemical potential for a higher spin current.
- This ensures the vanishing of the relative entropy which is expected to be true for any quantum mechanical system.
- The relative entropy in (1+1)d is independent of the UV cut-off. It's a refined observable in this sense.
- We have also probed the short-distance behaviour of the holographic EE and seen that it has the desired behaviour.
- All this lends strong support in favour of the holomorphic-Wilson line functional as the bulk observable which captures entanglement entropy.

Summary & Outlook

To summarize ...

- We have evaluated the entanglement entropy in CFTs with $\mathcal{W}$-symmetries deformed by a chemical potential for a higher-spin current.
- Computations were initially done using free field theories which have a higher-spin symmetry algebra.
- The first correction due to non-zero chemical potential is universal. This was proved using OPEs and uniformization techiniques.
- This universality was also confirmed by the Wilson line functional for holographic entanglement entropy.
- We also investigated relative entropy in this context. We found from holography that it has the expected behaviour in the short distance regime.

Ongoing work

- Analysing entropies of free fields on the torus with chemical potentials (modular forms, elliptic functions ...).
- How can the replica trick/uniformization be realized in the dual Chern-Simons language?

Thank you.

Backup slides

A bit about $\mathcal{W}$ algebra CFTs

- The symmetry algebra of 1+1-dimensional CFTs is that of two copies of the Virasoro algebra. This symmetry is infinite-dimensional.
  [Belavin-Polyakov-Zamolodchikov '84]

- One may consider studying higher-spin extensions of the Virasoro algebra – $\mathcal{W}_N$ algebras. [Zamolodchikov '85]

  These non-trivial extensions play a role in analysis and classification of CFTs and also appear as scaling limits of $\mathbb{Z}_N$ lattice models.

  These are also the symmetries of coset-WZW models, massless free fermions/bosons, RCFTs etc.

A bit about $\mathcal{W}$ algebra CFTs

Example

The commutation relations for the $\mathcal{W}_3$ algebra are

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \quad \text{(Virasoro sub-algebra)}$$

$$[L_n, W_m] = (2n-m)W_{n+m}$$

$$[W_n, W_m] = (n-m)\left[\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right]L_{n+m} + \frac{16}{5c+22}(n-m)\Lambda^{(4)}_{n+m} + \frac{c}{360}n(n^2-4)(n^2-1)\delta_{n+m,0}$$

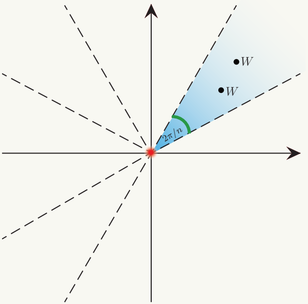
The global part at $c \to \infty$ or the wedge subalgebra is that of $sl(3,\mathbb{R})$.
W_m are Laurent modes of the (3,0) primary operator.

Method of images

The correlator $\langle WW \rangle$ on $\mathcal{R}_n$ can also be obtained by direct conformal transformation of the 2-point function on the plane.

Upon the uniformization transformation $w_p = \left(\frac{z-y_2}{z-y_1}\right)^{1/n} e^{2\pi i p/n}$, the pth replica gets mapped to a sector on the complex plane.

This is another well-known way of performing from the replica trick by introducing a conical singularity at the origin.



We then need to sum over the images since $W = \sum_{p=0}^{n-1} W_p$.

It can be seen that the same answer for $\langle W(z_1)W(z_2)\rangle_{\mathcal{R}_n}$ is reproduced.
[Long '14]

Regularization prescription

- We indeed encounter improper integrals while doing conformal perturbation theory.
- A principal value prescription is chosen to regulate these integrals

$$\int_0^{i\beta} d\tau_2 \int_{-\infty}^{\infty} d\sigma_2 \int_0^{i\beta} d\tau_1 \int_{-\infty}^{\infty} d\sigma_1 \, \langle W(z_1) W(z_2) \rangle$$

- The spatial integrals are then performed first

$$\int_0^{i\beta} d\tau_2 \left(\int_0^{\tau_2 - \epsilon} + \int_{\tau_2 + \epsilon}^{i\beta} \right) d\tau_1 \, \langle Q(\tau_1) Q(\tau_2) \rangle \; \rightarrow \; \beta^2 \langle Q^2 \rangle$$

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A thermodynamic relation for relative entropy

$$
\begin{aligned}
S(\sigma||\rho) &= \mathrm{tr}(\sigma \ln \sigma) - \mathrm{tr}(\sigma \ln \rho) \\
&= \mathrm{tr}(\sigma \ln \sigma) - \mathrm{tr}(\rho \ln \rho) + \mathrm{tr}(\rho \ln \rho) - \mathrm{tr}(\sigma \ln \rho) \\
&= -S_\sigma + S_\rho - \mathrm{tr}(\rho H_\rho) + \mathrm{tr}(\sigma H_\rho) \\
&= \Big(\big\langle H \big\rangle_\sigma - \big\langle H \big\rangle_\rho \Big) - (S_\sigma - S_\rho) \\
&= \Delta \big\langle H \big\rangle - \Delta S
\end{aligned}
$$